1. Mathematical Induction and Combinatorics

(1) Show that for each positive integer n, we have

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

(2) Show that the cube of a positive integer can always be written as the difference of two squares.

(3) Establish a formula for $\displaystyle\sum_{k=2}^{n} \frac{1}{k^2-1}$ valid for each positive integer n.

(4) Establish a formula allowing one to obtain the sum of the first n positive even integers.

(5) Show that the formula $\displaystyle\sum_{j=1}^{n} (-1)^j j^2 = (-1)^n \sum_{j=1}^{n} j$ holds for each positive integer n.

(6) Show that $a+b$ is a factor of $a^{2n-1}+b^{2n-1}$ for each integer $n \geq 1$.

(7) Show that a^2+b^2 is a factor of $a^{4n}-b^{4n}$ for each integer $n \geq 1$.

(8) Show that for each positive integer n,

$$a^n - b^n = (a-b)\sum_{k=0}^{n-1} a^k b^{n-1-k}.$$

(9) Show that $\displaystyle\sum_{j=1}^{n} j \cdot j! = (n+1)! - 1$ for each positive integer n.

(10) Prove, using induction, that $(2n)! < 2^{2n}(n!)^2$ for each integer $n \geq 1$.

(11) Use induction in order to prove that $n^3 < n!$ for each integer $n \geq 6$.

(12) Let θ be a real number such that $\theta \geq -1$. Prove, using induction, that for each integer $n \geq 0$, we have $(1+\theta)^n \geq 1 + n\theta$.

(13) Let θ be a nonnegative real number. Show, using induction, that for each positive integer n, we have $(1+\theta)^n \geq 1 + n\theta + \frac{n(n-1)}{2}\theta^2$.

(14) Show that for each positive integer n, $\frac{1}{3}(n^3+2n)$ is an integer.

(15) Show that $\dfrac{10^n + 3 \cdot 4^{n+2} + 5}{9}$ is an integer for each positive integer n.

(16) Show that if n is a positive integer, then

$$\binom{n}{k} = \binom{n}{k+1} \iff n = 2k+1.$$

(17) Show that if n is a positive integer, then

(a) $\displaystyle\binom{2n}{0} + \binom{2n}{2} + \binom{2n}{4} + \cdots + \binom{2n}{2n} = 2^{2n-1};$

(b) $\displaystyle\binom{2n}{1} + \binom{2n}{3} + \cdots + \binom{2n}{2n-1} = 2^{2n-1}.$

(18) Prove that for each integer $n \geq 1$, we have

$$n! \leq \left(\frac{n+1}{2}\right)^n.$$

(19) Show that each integer $n > 7$ can be written as a sum containing only the numbers 3 and 5. For example, $8 = 3 + 5$, $9 = 3 + 3 + 3$, $10 = 5 + 5$.

(20) Assume that amongst n points, $n \geq 2$, in a given plane, no three points are on the same line. Show that the number of possible lines passing through these points is $n(n-1)/2$.

(21) Show that for each integer $n \geq 2$,

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}.$$

(22) Prove that for each positive integer k,

$$1^3 + 3^3 + 5^3 + \cdots + (2k-1)^3 = k^2(2k^2 - 1).$$

(23) We saw in problem 1 that, for each integer $n \geq 1$,

$$\begin{aligned} 1 + 2 + \cdots + n &= \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}; \\ 1^2 + 2^2 + \cdots + n^2 &= \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}; \\ 1^3 + 2^3 + \cdots + n^3 &= \frac{n^2(n+1)^2}{4} = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}. \end{aligned}$$

Hence, letting $S_k(n) = 1^k + 2^k + \cdots + n^k$ and in light of these three relations, it is normal to conjecture that, for each integer $k \geq 1$, $S_k(n)$ is a polynomial of degree $k+1$. In fact, in 1654, Blaise Pascal (1623–1662) established that indeed it was the case. His proof used induction and the expansion of the expression $(n+1)^{k+1} - 1$. Provide the details.

(24) Find a formula, valid for each integer $n \geq 2$, for

$$\prod_{i=2}^{n} \left(1 - \frac{1}{i}\right), \quad \text{and the same for} \quad \prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right).$$

(25) Show that, whatever the value of the integer $n \geq 1$, we always have

$$\sum_{i=1}^{n} \frac{i}{i^4 + i^2 + 1} < \frac{1}{2}.$$

(26) Show that if m, n and r are three positive integers such that

$$S := \frac{1}{m} + \frac{1}{n} + \frac{1}{r} < 1, \quad \text{then} \quad S \leq \frac{41}{42}.$$

(27) Given a positive integer n, let $s(n)$ be the sum of its digits (in basis 10). For each pair of positive integers k, ℓ smaller than 10, let $A_k(\ell)$ be the number of ℓ-digit positive integers n whose sum of digits is equal to k. In other words,

$$A_k(\ell) = \#\{n : 10^{\ell-1} \leq n < 10^\ell,\ s(n) = k\}.$$

Show that

$$A_k(\ell) = \binom{k+\ell-2}{k-1} = \binom{k+\ell-2}{\ell-1},$$

and conclude in particular that $A_k(\ell) = A_\ell(k)$.

(28) Using induction, prove the formulas due to Mariares (1913):

$$\begin{aligned} 1^2 + 3^2 + 5^2 + \cdots + n^2 &= \binom{n+2}{3}, \quad \text{if } n \text{ is odd;} \\ 2^2 + 4^2 + 6^2 + \cdots + n^2 &= \binom{n+2}{3}, \quad \text{if } n \text{ is even.} \end{aligned}$$

(29) Let S be a set of 10 distinct integers chosen amongst the numbers $1, 2, \ldots,$ 99. Show that S must contain two disjoint subsets for which the sum of their respective elements is the same.

(30) Given 51 arbitrary positive integers, show that one can always find two of them whose difference is 50.

(31) In order to acquire problem solving skills, a student decides to solve at least one problem per day and at most 11 per week and to do this for a whole year. Show that there exists a period of consecutive days during which he will solve exactly 20 problems.

(32) On a rectangular table of dimension 120 inches by 150 inches, we set 14 001 marbles. Show that, no matter how these are arranged, one can place a cylindrical glass with a diameter of 5 inches over at least 8 marbles.

(33) Choose n points on a circle and join them pairwise by secants. Taking for granted that no more than two secants can meet at the same point, in how many regions is the circle thus divided?

(34) Say we have three posts and n disks of different diameters placed on one of the posts, ordered by increasing diameters, the largest at the bottom of the post, the smaller at the top. The problem consists in transferring the tower of disks from the first post to the third post, using if need be the second post, but in such a way that, with each move, we do not place the moving disk on a smaller one. Establish the function of n which gives the minimum number of moves. (This problem is known as the "Tower of Hanoi Problem".)

(35) Let $\{F_n : n \in \mathbb{N}\}$ be the sequence of Fibonacci numbers defined by $F_1 = 1, F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Show that each positive integer can be written as the sum of distinct Fibonacci numbers.

(36) One easily checks that

$$\begin{aligned} 1 &= 1^2, \\ 2 &= -1^2 - 2^2 - 3^2 + 4^2, \\ 3 &= -1^2 + 2^2, \\ 4 &= -1^2 - 2^2 + 3^2, \\ 5 &= 1^2 + 2^2, \\ 6 &= 1^2 - 2^2 + 3^2. \end{aligned}$$

Hence, we may be tempted to formulate a conjecture, namely that each positive integer n can be written as

$$n = e_1 1^2 + e_2 2^2 + e_3 3^2 + e_4 4^2 + \cdots + e_k k^2,$$

for a certain positive integer k (depending on n), where the $e_i \in \{-1, 1\}$. Prove this conjecture.

2. Divisibility

(37) The mathematician Duro Kurepa defined $!n = 0! + 1! + \cdots + (n-1)!$ for $n \geq 1$ and conjectured that $(!n, n!) = 2$ for all $n \geq 2$. This conjecture has been verified by Ivić and Mijajlović [**20**] for $n < 10^6$. Using computer software, write a program showing that this conjecture is true up to $n = 1000$.

(38) Consider the situation where the positive integer a is divided by the positive integer b using the euclidian division (see Theorem 7) yielding

$$(*) \qquad a = 652b + 8634.$$

By how much can we increase both a and b without changing the quotient $q = 652$?

(39) Consider the number $N = 111\ldots11$, here written in basis 2. Write N^2 in basis 2.

(40) Show that $39|7^{37} + 13^{37} + 19^{37}$.

(41) Show that, for each integer $n \geq 1$, the number $49^n - 2352n - 1$ is divisible by 2304.

(42) Given any integer $n \geq 1$, show that the number $n^4 + 2n^3 + 2n^2 + 2n + 1$ is never a perfect square.

(43) Let N be a two digit number. Let M be the number obtained from N by interchanging its two digits. Show that 9 divides $M - N$ and then find all the integers N such that $|M - N| = 18$.

(44) Is it true that 3 never divides $n^2 + 1$ for every positive integer n? Explain.

(45) Is it true that 5 never divides $n^2 + 2$ for every positive integer n? Explain. Is the result the same if one replaces the number 5 by the number 7?

(46) Given $s + 1$ integers $a_0, a_1, \ldots, a_s$ and a prime number p, show that p divides the integer

$$N(n) := a_0 + a_1 n + \cdots + a_{s-1} n^{s-1} + a_s n^s$$

if and only if p divides $N(r)$, for an integer r, $0 \leq r \leq p - 1$. Use this to find all integers n such that 7 divides $3n^2 + 6n + 5$.

(47) Compute the value of the expression

$$\frac{(10^4+324)(22^4+324)(34^4+324)(46^4+324)(58^4+324)}{(4^4+324)(16^4+324)(28^4+324)(40^4+324)(52^4+324)}.$$

(48) Show that, in any basis, the number 10101 is composite.

(49) Show that the product of four consecutive integers is necessarily divisible by 24.

(50) Show that the number

$$1^{47} + 2^{47} + 3^{47} + 4^{47} + 5^{47} + 6^{47}$$

is a multiple of 7.

(51) Show that the product of any five consecutive positive integers cannot be a perfect square.

(52) Show that $30|n^5 - n$ for each positive integer n.

(53) Show that $6|n(n+1)(2n+1)$ for each positive integer n.

(54) Given any integer $n \geq 0$, show that $64^{n+1} - 63n - 64$ is divisible by 3969. More generally, given $a \in \mathbb{N}$, show that for each integer $n \geq 0$, $(a+1)^{n+1} - an - (a+1)$ is divisible by a^2.

(55) Find all positive integers n such that $(n+1)|(n^2+1)$.

(56) Find all positive integers n such that $(n^2+2)|(n^6+206)$.

(57) Identify, if any exist, the positive integers n such that $(n^3+2)|(n^6+216)$.

(58) If a and b are positive integers such that $b|(a^2+1)$, do we necessarily have that $b|(a^4+1)$? Explain.

(59) Let n and k be positive integers.

(a) For $n \geq k$, show that

$$\frac{n}{(n,k)} \;\Big|\; \binom{n}{k}.$$

(b) For $n \geq k$, show that

$$\frac{n+1-k}{(n+1,k)} \;\Big|\; \binom{n}{k}.$$

(c) For $n \geq k-1 \geq 1$, show that

$$\frac{(n+1,k-1)}{n+2-k}\binom{n}{k-1} \quad \text{is an integer.}$$

(60) For each integer $n \geq 1$, let $f(n) = 1! + 2! + \cdots + n!$. Find polynomials $P(x)$ and $Q(x)$ such that

$$f(n+2) = P(n)f(n+1) + Q(n)f(n), \qquad \text{for each integer } n \geq 1.$$

(61) Show that, for each positive integer n,

$$49|2^{3n+3} - 7n - 8.$$

(62) Find all positive integers a for which $a^{10}+1$ is divisible by 10.

(63) Is it true that $3|2^{2n} - 1$ for each positive integer n? Explain.

(64) Show that if an integer is of the form $6k+5$, then it is necessarily of the form $3k-1$, while the reverse is false.

(65) Can an integer $n > 1$ be of the form $8k+7$ and also of the form $6\ell+5$? Explain.

(66) Let $M_1 = 2+1$, $M_2 = 2\cdot 3+1$, $M_3 = 2\cdot 3\cdot 5+1$, $M_4 = 2\cdot 3\cdot 5\cdot 7+1$, $M_5 = 2\cdot 3\cdot 5\cdot 7\cdot 11+1, \ldots$. Prove none of the numbers M_k is a perfect square.

(67) Verify that if an integer is a square and a cube, then it must be of the form $7k$ or $7k+1$.

(68) If x and y are odd integers, prove that x^2+y^2 cannot be a perfect square.

(69) Show that, for each positive integer n, we have $n^2|(n+1)^n - 1$.

(70) Let $k, n \in \mathbb{N}$, $n \geq 2$. Show that $(n-1)^2|(n^k-1)$ if and only if $(n-1)|k$. More generally, show the following result: Let $a \in \mathbb{Z}$ and $k, n \in \mathbb{N}$ with $n \neq a$; then $(n-a)^2|(n^k - a^k)$ if and only if $(n-a)|ka^{k-1}$.

(71) Let a, b be integers and let n be a positive integer.

(a) If $a - b \neq 0$, show that

$$\left(\frac{a^n - b^n}{a-b}, a-b\right) = \left(n(a,b)^{n-1}, a-b\right).$$

(b) If $a+b \neq 0$ and if n is odd, show that

$$\left(\frac{a^n+b^n}{a+b}, a+b\right) = \left(n(a,b)^{n-1}, a+b\right).$$

(c) Show that if a and b are relatively prime with $a+b \neq 0$ and if $p > 2$ is a prime number, then

$$\left(\frac{a^p+b^p}{a+b}, a+b\right) = \begin{cases} 1 & \text{if } p \not| (a+b), \\ p & \text{if } p|(a+b). \end{cases}$$

(72) Let k and n be positive integers. Show that the only solutions (k,n) of the equation $(n-1)! = n^k - 1$ are $(1,2)$, $(1,3)$ and $(2,5)$.

(73) According to Euclid's algorithm, assuming that $b \geq a$ are positive integers, we have

$$\begin{array}{rcll} b &=& aq_1 + r_1, & 0 < r_1 < a, \\ a &=& r_1 q_2 + r_2, & 0 < r_2 < r_1, \\ r_1 &=& r_2 q_3 + r_3, & 0 < r_3 < r_2, \\ & \vdots & & \\ r_{j-2} &=& r_{j-1} q_j + r_j, & 0 < r_j < r_{j-1}, \\ r_{j-1} &=& r_j q_{j+1}, & \end{array}$$

where $r_j = (a,b)$.

(a) Show that $b > 2r_1$, $a > 2r_2$ and for $k \geq 1$, $r_k > 2r_{k+2}$.

(b) Deduce that $b > 2^{j/2}$ and therefore that the maximum number of steps in Euclid's algorithm is $[2(\log b / \log 2)]$.

(74) Show that there exist infinitely many positive integers n such that $n|2^n+1$.

(75) Let a be an integer ≥ 2. Show that for positive integers m and n we have

$$a^n - 1 | a^m - 1 \iff n|m.$$

(76) Let N_n be an integer formed of n consecutive "1"s. For example, $N_3 = 111$, $N_7 = 1\,111\,111$. Show that $N_n | N_m \iff n|m$.

(77) Prove that no member of the sequence $11, 111, 1\,111, 11\,111, \ldots$ is a perfect square.

(78) What is the smallest positive integer divisible both by 2 and 3 which is both a perfect square and a sixth power? More generally, what is the smallest positive integer n divisible by both 2 and 3 which is both an n–th power and an m–th power, where $n, m \geq 2$?

(79) Three of the four integers, found between 100 and 1000, with the property of being equal to the sum of the cubes of their digits are 153, 370 and 407. What is the fourth of these integers?

(80) How many positive integers $n \leq 1000$ are not divisible by 2, nor by 3, nor by 5?

(81) Prove the following result obtained in the seventeenth century by Pierre de Fermat (1601–1665): *"Each odd prime number p can be written as the difference of two perfect squares."*

(82) Prove that the representation mentioned in problem 81 is unique.

(83) Is the result of Fermat stated in problem 81 still true if p is simply an odd positive integer?

(84) Let $n = 999\,980\,317$. Observing that $n = 10^9 - 3^9$ and factoring this last expression, conclude that $7|n$.

(85) Show that if an odd integer can be written as the sum of two squares, then it is of the form $4n+1$.

(86) Let $a,b,c \in \mathbb{Z}$ be such that $abc \neq 0$ and $(a,b,c) = 1$ and such that $a^2+b^2=c^2$. Prove that at least one of the integers a and b is even.

(87) For which integer values of k is the number $10^k - 1$ the cube of an integer?

(88) Show that if the positive integer a divides both $42n+37$ and $7n+4$ for a certain integer n, then $a=1$ or $a=13$.

(89) If a and b are two positive integers and if $\dfrac{1}{a}+\dfrac{1}{b}$ is an integer, prove that $a=b$. Moreover, show that a is then necessarily equal to 1 or 2.

(90) Let $a,b \in \mathbb{N}$ such that $(a,b)=4$. Find all possible values of (a^2,b^3).

(91) Let $a,b \in \mathbb{N}$ and $d=(a,b)$. Find the value of $(3a+5b, 5a+8b)$ in terms of d and more generally that of $(ma+nb, ra+sb)$ knowing that $ms-nr=1$, where $m,n,r,s \in \mathbb{N}$.

(92) Let $m,n \in \mathbb{N}$. If $d|mn$ where $(m,n)=1$, show that d can be written as $d=rs$ where $r|m$, $s|n$ and $(r,s)=1$.

(93) Let a,b,d be nonzero integers, d odd, such that $d|(a+b)$ and $d|(a-b)$. Show that $d|(a,b)$.

(94) Given eight positive composite integers ≤ 360, show that at least two of them have a common factor larger than 1.

(95) If a and b are positive integers such that $(a,b)=1$ and ab is a perfect square, show that a and b are perfect squares.

(96) Can $n(n+1)$ be a perfect square for a certain positive integer n? Explain.

(97) What are the possible values of the expression $(n, n+14)$ as n runs through the set of positive integers?

(98) Let $n>1$ an integer. Which of the following statements are true:

$$3|(n^3-n), \qquad 3|n(n+1), \qquad 8|(2n+1)^2-1, \qquad 6|n(n+1)(n+2).$$

(99) Is it true that if n is an even integer, then $24|n(n+1)(n+2)$? Explain.

(100) Let n be an integer such that $(n,2)=(n,3)=1$. Show that $24|n^2+47$.

(101) Let $d=(a,b)$, where a and b are positive integers. Show that there are exactly d numbers amongst the integers a, $2a$, $3a, \ldots, ba$ which are divisible by b.

(102) Let a,b be integers such that $(a,b)=d$, and let x_0,y_0 be integers such that $ax_0+by_0=d$. Show that:

(a) $(x_0,y_0)=1$;

(b) x_0 and y_0 are not unique.

(103) Let a,m and n be positive integers. If $(m,n)=1$, show that $(a,mn) = (a,m)(a,n)$.

(104) For all $n \in \mathbb{N}$, show that $(n^2+3n+2, 6n^3+15n^2+3n-7)=1$.

(105) Let $a,b \in \mathbb{Z}$. If $(a,b)=1$, show that

(a) $(a+b,a-b)=1$ or 2; (b) $(2a+b,a+2b)=1$ or 3;

(c) $(a^2+b^2,a+b)=1$ or 2; (d) $(a+b,a^2-3ab+b^2)=1$ or 5.

(106) Let $a,b \in \mathbb{Z}$. If $(a,b)=1$, find the possible values of

(a) (a^3+b^3,a^3-b^3); (b) (a^2-b^2,a^3-b^3).

(107) Let a, b and c be integers. For each of the following statements, say if it is true or false. If it is true, give a proof; if it is false, provide a counter-example.

(a) If $(a,b)=(a,c)$, then $[a,b]=[a,c]$.

(b) If $(a,b)=(a,c)$, then $(a^2,b^2)=(a^2,c^2)$.
(c) If $(a,b)=(a,c)$, then $(a,b)=(a,b,c)$.

(108) Let $a,b\in\mathbb{Z}$ and let $m,n\in\mathbb{N}$. For each of the following statements, say if it is true or false. If it is true, give a proof; if it is false, provide a counter-example.
(a) If $a^n|b^n$, then $a|b$.
(b) If $a^m|b^n$, $m>n$, then $a|b$.
(c) If $a^m|b^n$, $m<n$, then $a|b$.

(109) Let $a,b,c\in\mathbb{Z}$. Show that if $(a,b)=1$ and $c|a$, then $(c,b)=1$.

(110) Let $a,b,c\in\mathbb{Z}$. Show that if $(a,bc)=1$, then $(a,b)=(a,c)=1$.

(111) Let $a,b\in\mathbb{Z}$. Show that $(a,b)=(a+b,[a,b])$. Using this result, find two positive integers whose sum is 186 and whose LCM is 1440.

(112) Let $a,b,c\in\mathbb{Z}$.
(a) Show that $(a,bc)=(a,(a,b)c)$.
(b) Show that $(a,bc)=(a,(a,b)(a,c))$.

(113) Let $a,b,c\in\mathbb{Z}$. Show that if $(a,c)=1$, then $(ab,c)=(b,c)$.

(114) Let a, b, m and n be integers. If $(m,n)=1$, show that $(ma+nb,mn)=(a,n)(b,m)$. Show that this result generalizes the result of problem 103.

(115) Is it possible that $\binom{n}{r}$ is relatively prime with $\binom{n}{s}$, for certain positive integers r,s,n satisfying $0<r<s\le n/2$? Explain.

(116) Find two positive integers for which the difference between their LCM and their GCD is equal to 143.

(117) Let a,b,c be positive integers. Show that $(a,b,c)=((a,b),c)$ and $[a,b,c]=[[a,b],c]$. Generalize this result. Use this result to compute $(132,102,36)$ and find those integers x,y,z for which $132x+102y+36z=(132,102,36)$.

(118) Let n be a positive integer. Evaluate $(n,n+1,n+2)$ and $[n,n+1,n+2]$.

(119) Let a,b,c be positive integers. If $(a,b)=(b,c)=(a,c)=1$, show that $(a,b,c)[a,b,c]=abc$.

(120) Is it true that if a and b are positive integers such that $(a,b)=1$, then $(a^2,ab,b^2)=1$? Explain.

(121) Is it true that if a, b and c are positive integers, then $[a^2,ab,b^2]=[a^2,b^2]$? Explain.

(122) Is it true that if a, b and c are positive integers, then $(a,b,c)=\big((a,b),(a,c)\big)$? Explain.

(123) Is it true that $[a,b,c]\cdot(a,b,c)=|abc|$, $\forall\ a,b,c\in\mathbb{Z}\setminus\{0\}$? Explain.

(124) Let a, b, d, m and n be positive integers such that $a|d^m-1$, $b|d^n-1$ and $(a,b)=1$. Show that $ab|d^{[m,n]}-1$.

(125) Show that if a is an integer >1, then, for each pair of positive integers m and n,

$$(a^m-1,a^n-1)=a^{(m,n)}-1.$$

What do we obtain for (a^m+1,a^n+1), for (a^m+1,a^n-1)? More generally, given $a>1$ and $b>1$, what are the values of

$$(a^m-b^m,a^n-b^n),\quad (a^m+b^m,a^n+b^n)\ \text{ and }\ (a^m+b^m,a^n-b^n)?$$

(126) Show that there exist infinitely many pairs of integers $\{x,y\}$ satisfying $x+y=40$ and $(x,y)=5$.

(127) Find all pairs of positive integers $\{a, b\}$ such that $(a, b) = 15$ and $[a, b] = 90$. More generally, if d and m are positive integers, show that there exists a pair of positive integers $\{a, b\}$ for which $(a, b) = d$ and $[a, b] = m$ if and only if $d|m$. Moreover, in this situation, show that the number of such pairs is 2^r, where r is the number of distinct prime factors of m/d.

(128) Prove that one cannot find integers m and n such that $m + n = 101$ and $(m, n) = 3$.

(129) Let $a, m, n \in \mathbb{N}$ with $m \neq n$.
(a) Show that $(a^{2^n} + 1)|(a^{2^m} - 1)$ if $m > n$.
(b) Show that $(a^{2^n} + 1, a^{2^m} + 1) = \begin{cases} 1 & \text{if } a \text{ is even,} \\ 2 & \text{if } a \text{ is odd.} \end{cases}$

(130) Let n be a positive integer. Find the greatest common divisor of the numbers
$$\binom{2n}{1}, \binom{2n}{3}, \binom{2n}{5}, \ldots, \binom{2n}{2n-1}.$$

(131) Given $n + 1$ distinct positive integers $a_1, a_2, \ldots, a_{n+1}$ such that $a_i \leq 2n$ for $i = 1, 2, \ldots, n + 1$, show that there exists at least one pair $\{a_j, a_k\}$ with $j \neq k$ such that $a_j | a_k$.

(132) Let $n > 2$. Consider the three n-tuples $(a_1^{(i)}, a_2^{(i)}, \ldots, a_n^{(i)})$, $i = 1, 2, 3$, where each $a_j^{(i)} \in \{+1, -1\}$ and assume that these three n-tuples satisfy $\displaystyle\sum_{j=1}^{n} a_j^{(i)} a_j^{(k)} = 0$ for each pair $\{i, k\}$ such that $1 \leq i < k \leq 3$. Show that $4|n$.

(133) Let A be the set of natural numbers which, in their decimal representation, do not have "7" amongst their digits. Prove that
$$\sum_{n \in A} \frac{1}{n} < +\infty.$$

(134) Let $u_1, u_2, \ldots$ be a strictly increasing sequence of positive integers. Denoting by $[a, b]$ the lowest common multiple of a and b, show that the series
$$\sum_{n=1}^{\infty} \frac{1}{[u_n, u_{n+1}]} \qquad \text{converges.}$$

3. Prime Numbers

(135) Using computer software, write a program
  (a) to generate all Mersenne primes up to $2^{525} - 1$;
  (b) to determine the smallest prime number larger than $10^{100} + 1$.

(136) Write a program that generates prime numbers up to a given number N. One can, of course, use Eratosthenes' sieve.

(137) Use a computer to find four consecutive integers having the same number of prime factors (allowing repetitions).

(138) (a) By reversing the digits of the prime number 1009, we obtain the number 9001, which is also prime. Write a program to find the prime numbers in [1,10000] verifying this property.
(b) By reversing the digits of the prime number 163, we obtain the number 361, which is a perfect square. Using computer software, write a program to find all prime numbers in $[1, 10000]$ with this property.

(139) Using a computer, find all prime numbers $p \leq 10\,000$ with the property that p, $p+2$ and $p+6$ are all primes.

(140) Let p_k be the k-th prime number. Show that $p_k < 2^k$ if $k \geq 2$.

(141) If a prime number $p_k > 5$ is equally isolated from the prime numbers appearing before and after it, that is $p_k - p_{k-1} = p_{k+1} - p_k = d$, say, show that d is a multiple of 6. Then, for each of the cases $d = 6$, 12 and 18, find, by using a computer, the smallest prime number p_k with this property.

(142) Prove that none of the numbers

$$12321,\ 1234321,\ 123454321,\ 12345654321,\ 1234567654321,$$
$$123456787654321,\ 12345678987654321$$

is prime.

(143) For each integer $k \geq 1$, let n_k be the k-th composite number, so that for instance $n_1 = 4$ and $n_{10} = 18$. Use computer software and an appropriate algorithm in order to establish the value of n_k, with $k = 10^\alpha$, for each integer $\alpha \in [2, 10]$.

(144) For each integer $k \geq 1$, let n_k be the k-th number of the form p^α, where p is prime, α a positive integer, so that for instance $n_1 = 2$ and $n_{10} = 16$. Use computer software and an appropriate algorithm in order to establish the value of n_k, with $k = 10^\alpha$, for each integer $\alpha \in [2, 10]$.

(145) Find all positive integers $n < 100$ such that $2^n + n^2$ is prime. To which class of congruence modulo 6 do these numbers n belong?

(146) Show that if the integer $n \geq 4$ is not an odd multiple of 9, then the corresponding number $a_n := 4^n + 2^n + 1$ is necessarily composite. Then, use a computer in order to find all positive integers $n < 1000$ for which a_n is prime.

(147) Consider the sequence (a_n) defined by $a_1 = a_2 = 1$ and, for $n \geq 3$, by $a_n = n! - (n-1)! + \cdots + (-1)^n 2! + (-1)^{n+1} 1!$. Use a computer in order to find the smallest number n such that a_n is a composite number.

(148) The mathematicians Minác and Willans have obtained a formula for the n-th prime number p_n which is more of a theoretical interest than of a

practical interest:

$$p_n = 1 + \sum_{m=1}^{2^n} \left[\left[\frac{n}{1 + \sum_{j=2}^{m} \left[\frac{(j-1)!+1}{j} - \left[\frac{(j-1)!}{j} \right] \right]} \right]^{1/n} \right],$$

where as usual $[x]$ stands for the largest integer $\leq x$. Prove this formula.

(149) Develop an idea used by Paul Erdős (1913–1996) to show that, for each integer $n \geq 1$,

$$\prod_{p \leq n} p \leq 4^n.$$

His idea was to write

$$\prod_{p \leq n} p = \prod_{p \leq \frac{n+1}{2}} p \cdot \prod_{\frac{n+1}{2} < p \leq n} p$$

and to use the fact that each prime number $p > (n+1)/2$ appears in the factorization of the binomial coefficient $\binom{n}{(n+1)/2}$. Provide the details.

(150) Show that if four positive integers a, b, c, d are such that $ab = cd$, then the number $a^2 + b^2 + c^2 + d^2$ is necessarily composite.

(151) Show that, for each integer $n \geq 1$, the number $4n^3 + 6n^2 + 4n + 1$ is composite.

(152) Show that if p and q are two consecutive odd prime numbers, then $p + q$ is the product of at least three prime numbers (not necessarily distinct).

(153) Does there exist a positive integer n such that $n/2$ is a perfect square, $n/3$ a cube and $n/5$ a fifth power?

(154) Given any integer $n \geq 2$, show that $n^{42} - 27$ is never a prime number.

(155) Let $\theta(x) := \sum_{p \leq x} \log p$. Prove that Bertrand's Postulate follows from the fact that

$$c_1 x < \theta(x) < c_2 x,$$

where $c_1 = 0.73$ and $c_2 = 1.12$.

(156) Use Bertrand's Postulate to show that, for each integer $n \geq 4$,

$$p_{n+1}^2 < p_1 p_2 \cdots p_n,$$

where p_n stands for the n-th prime number.

(157) Certain integers $n \geq 3$ can be written in the form $n = p + m^2$, with p prime and $m \in \mathbb{N}$. This is the case for example for the numbers 3, 4, 6, 7, 8, 9, 11, 12, 14, 15, 16, 17, 18, 19, 20, 21. Let q^r be a prime power, where r is a positive even integer such that $2q^{r/2} - 1$ is composite. Show that q^r cannot be written as $q^r = p + m^2$, with p prime and $m \in \mathbb{N}$.

(158) Show that if p and $8p - 1$ are primes, then $8p + 1$ is composite.

(159) Show that all positive integers of the form $3k + 2$ have a prime factor of the same form, that all positive integers of the form $4k + 3$ have a prime factor of the same form, and finally that all positive integers of the form $6k + 5$ have a prime factor of the same form.

(160) A positive integer n has a *Cantor expansion* if it can be written as

$$n = a_m m! + a_{m-1}(m-1)! + \cdots + a_2 2! + a_1 1!,$$

where the a_j's are integers satisfying $0 \leq a_j \leq j$.

(a) Find the Cantor expansion of 23 and of 57.

(b) Show that all positive integers n have a Cantor expansion and moreover that this expansion is unique.

(161) If $p > 1$ and $d > 0$ are integers, show that p and $p+d$ are both primes if and only if

$$(p-1)!\left(\frac{1}{p}+\frac{(-1)^d d!}{p+d}\right)+\frac{1}{p}+\frac{1}{p+d}$$

is an integer.

(162) Find all prime numbers p such that $p+2$ and p^2+2p-8 are primes.

(163) Is it true that if p and p^2+8 are primes, then p^3+4 is prime? Explain.

(164) Let $n \geq 2$. Show that the integers n and $n+2$ form a pair of twin primes if and only if

$$4\left((n-1)!+1\right)+n \equiv 0 \pmod{n(n+2)}.$$

(165) Identify each prime number p such that 2^p+p^2 is also prime.

(166) For which prime number(s) p is $17p+1$ a perfect square?

(167) Given two integers a and b such that $(a,b)=p$, where p is prime, find all possible values of:

(a) (a^2,b); (b) (a^2,b^2); (c) (a^3,b); (d) (a^3,b^2).

(168) Given two integers a and b such that $(a,p^2)=p$ and $(b,p^4)=p^2$, where p is prime, find all possible values of:

(a) (ab,p^5); (b) $(a+b,p^4)$; (c) $(a-b,p^5)$; (d) $(pa-b,p^5)$.

(169) Given two integers a and b such that $(a,p^2)=p$ and $(b,p^3)=p^2$, where p is a prime number, evaluate the expressions (a^2b^2,p^4) and (a^2+b^2,p^4).

(170) Let p be a prime number and a, b, c be positive integers. For each of the following statements, say if is true or false. If it is true, give a proof; if it is false, provide a counter-example.

(a) If $p|a$ and $p|(a^2+b^2)$, then $p|b$.
(b) If $p|a^n$, $n \geq 1$, then $p|a$.
(c) If $p|(a^2+b^2)$ and $p|(b^2+c^2)$, then $p|(a^2-c^2)$.
(d) If $p|(a^2+b^2)$ and $p|(b^2+c^2)$, then $p|(a^2+c^2)$.

(171) Let a, b and c be positive integers. Show that $abc=(a,b,c)[ab,bc,ac]=(ab,bc,ac)[a,b,c]$.

(172) Let a, b and c be positive integers and assume that $abc=(a,b,c)[a,b,c]$. Show that this necessarily implies that $(a,b)=(b,c)=(a,c)=1$.

(173) Let a, b and c be positive integers. Show that $(a,b,c)=\dfrac{(a,b)(b,c)(a,c)}{(ab,bc,ac)}$ and that $[a,b,c]=\dfrac{abc\,(a,b,c)}{(a,b)(b,c)(a,c)}$.

(174) Let a, b and c be positive integers. Show that

$$\frac{[a,b,c]^2}{[a,b][b,c][c,a]}=\frac{(a,b,c)^2}{(a,b)(b,c)(c,a)}.$$

(175) Find three positive integers a,b,c such that

$$[a,b,c]\cdot(a,b,c)=\sqrt{abc}.$$

(176) Let $\#n = [1, 2, 3, \dots, n]$ be the lowest common multiple of the numbers $1, 2, \dots, n$. Show that

$$\prod_{p \le n} p \le \#n = \prod_{p \le n} p^{[\log n / \log p]}.$$

(177) Let p be a prime number and r a positive integer. What are the possible values of $(p, p+r)$ and of $[p, p+r]$?

(178) Let $p > 2$ be a prime number such that $p | 8a - b$ and $p | 8c - d$, where $a, b, c, d \in \mathbb{Z}$. Show that $p | (ad - bc)$.

(179) Show that, if $\{p, p+2\}$ is a pair of twin primes with $p > 3$, then 12 divides the sum of these two numbers.

(180) Let n be a positive integer. Show that if n is a composite integer, then $n | (n-1)!$ except when $n = 4$.

(181) For which positive integers n is it true that

$$\sum_{j=1}^{n} j \;\Big|\; \prod_{j=1}^{n} j?$$

(182) Let $\pi = 3.141592\dots$ be Archimede's constant, and for each positive real number x, let $\pi_2(x)$ be the function that counts the number of pairs of twin primes $\{p, p+2\}$ such that $p \le x$. Show that

$$\pi_2(x) = 2 + \sum_{7 \le n \le x} \sin\left(\frac{\pi}{2}(n+2)\left[\frac{n!}{n+2}\right]\right) \cdot \sin\left(\frac{\pi}{2} n \left[\frac{(n-2)!}{n}\right]\right),$$

where $[y]$ stands for the largest integer $\le y$.

(183) Given an integer $n \ge 2$, show, without using Bertrand's Postulate, that there exists a prime number p such that $n < p < n!$.

(184) In 1556, Niccòlo Tartaglia (1500–1557) claimed that the sums

$$1+2+4,\ 1+2+4+8,\ 1+2+4+8+16,\ \dots$$

stood successively for a prime number and a composite number. Was he right?

(185) Show that if $a^n - 1$ is prime for certain integers $a > 1$ and $n > 1$, then $a = 2$ and n is prime.

REMARK: *The integers of the form $2^p - 1$, where p is prime, are called Mersenne numbers. We denote them by M_p in memory of Marin Mersenne* (1588–1648), *who had stated that M_p is prime for*

$$p = 2, 3, 5, 7, 13, 17, 19, 31, 67, 127, 257$$

and composite for all the other primes $p < 257$. This assertion of Mersenne can be found in the preface of his book Cogita Physico-mathematica, *published in Paris in 1644. Since then, we have found a few errors in the computations of Mersenne: indeed M_p is not prime for $p = 67$ and $p = 257$, while M_p is prime for $p = 61$, $p = 89$ and $p = 109$. One can find in the appendix C of the book of J.M. De Koninck and A. Mercier* [**8**] *the list of Mersenne primes M_p corresponding to the prime numbers p satisfying $2 \le p \le 44\,497$. Note on the other hand that it has recently been discovered that $2^{32\,582\,657} - 1$ is prime (in September 2006), which brings to 44 the total number of known Mersenne primes. It is also known that the primes*

M_p are closely related to the PERFECT NUMBERS, *in the sense that, as was shown by Leonhard Euler (1707–1783), n is an even perfect number if and only if $n = 2^{p-1}(2^p - 1)$, where $2^p - 1$ is a Mersenne prime.*

(186) Show that if there exists a positive integer n and an integer $a \geq 2$ such that $a^n + 1$ is prime, then a is even and $n = 2^r$ for a certain positive integer r.

REMARK: *The prime numbers of the form $2^{2^k} + 1$, $k = 0, 1, 2, \ldots$, are called "Fermat primes". The reason is that Pierre de Fermat claimed in 1640 (although saying he could not prove it) that all the numbers of the form $2^{2^k} + 1$ are prime. One hundred years later, Euler proved that*

$$2^{2^5} + 1 = 4294967297 = 641 \cdot 6700417.$$

As of today, we still do not know if, besides the cases $k = 0, 1, 2, 3, 4$, primes of the form $2^{2^k} + 1$ exist. Nevertheless, it is known that $2^{2^k} + 1$ is composite for $5 \leq k \leq 32$; see H.C. Williams [**41**] *and the site* www.prothsearch.net/fermat.html.

(187) Show that the equation $(2^x - 1)(2^y - 1) = 2^{2^z} + 1$ is impossible for positive integers x, y and z. (This implies in particular that a Fermat number, that is a number of the form $2^{2^k} + 1$, cannot be the product of two Mersenne numbers.)

(188) Prove by induction that, for each integer $n \geq 1$,

$$F_0 F_1 F_2 \cdots F_{n-1} = F_n - 2,$$

where $F_i = 2^{2^i} + 1$, $i = 0, 1, 2, \ldots$.

(189) Use the result of problem 188 in order to prove that if m and n are distinct positive integers, then $(F_m, F_n) = 1$.

(190) A positive integer n is said to be *pseudoprime in basis $a \geq 2$* if it is composite and if $a^{n-1} \equiv 1 \pmod{n}$. Find the smallest number which is pseudoprime in each of the bases 2, 3, 5 and 7.

(191) Use Problem 189 to prove that there exist infinitely many primes.

(192) Consider the numbers $f_n = 2^{3^n} + 1$, $n = 1, 2, \ldots$, and show they are all composite and in particular that, for each positive integer n,
(a) $3^{n+1} | f_n$; (b) $p | f_n \Rightarrow p | f_{n+1}$.

(193) Show that there exist infinitely many prime numbers p such that the numbers $p - 2$ and $p + 2$ are both composite.

(194) Show that 641 divides $F_5 = 2^{2^5} + 1$ without doing the explicit division.

(195) Use an induction argument in order to prove that each Fermat number $F_n = 2^{2^n} + 1$, where $n \geq 2$, ends with the digit 7.

(196) Let n be a positive integer and consider the set $E = \{1, 2, \ldots, n\}$. Let 2^k be the largest power of 2 which belongs to E. Show that for all $m \in E \setminus \{2^k\}$, we have $2^k \not| m$. Using this result, show that $\sum_{j=1}^{n} 1/j$ is not an integer if $n > 1$.

(197) Show that, for each positive integer n, one can find a prime number $p < 50$ such that $p | (2^{5n} - 1)$.

(198) Show that the integers defined by the sequence of numbers

$$M_k = p_1 p_2 \cdots p_k + 1 \qquad (k = 1, 2, \ldots),$$

where p_j stands for the j–th prime number, are prime numbers for $1 \leq k \leq 5$ and composite numbers for $k = 6, 7$. What about M_8, M_9 and M_{10}?

(199) Use the proof of Euclid's Theorem on the infinitude of primes to show that, if we denote by p_r the r-th prime number, then $p_r \leq 2^{2^{r-1}}$ for each $r \in \mathbb{N}$.

(200) In Problem 199, we obtained an upper bound for p_r, the r-th prime number, namely $p_r \leq 2^{2^{r-1}}$. Use this inequality to obtain a lower bound for $\pi(x)$, the number of prime numbers $\leq x$. More precisely, show that, for $x \geq 3$, $\pi(x) \geq \log \log x$.

(201) Show that there exist infinitely many prime numbers of the form $4n + 3$.

(202) Show that there exist infinitely many prime numbers of the form $6n + 5$.

(203) Let $f \colon \mathbb{N} \to \mathbb{R}$ be the function defined by

$$f(x) = a_r x^r + a_{r-1} x^{r-1} + \cdots + a_1 x + a_0,$$

where $a_r \neq 0$ and where each a_i, $0 \leq i \leq r$, is an integer. Show that, by an appropriate choice of a_i, $0 \leq i \leq r$, the set $\{f(n) : n \in \mathbb{N}\}$ contains at least r prime numbers.

(204) Consider the positive integers which can be written as an alternating sequence of 0's and 1's. The number 101 010 101 is such a number and observe that $101\,010\,101 = 41 \cdot 271 \cdot 9091$. Besides 101, do there exist other prime numbers of this form?

(205) Find all prime numbers of the form $2^{2^n} + 5$, where $n \in \mathbb{N}$. Would the question be more difficult if one replaces the number 5 by another number of the form $3k + 2$? Explain.

(206) The largest gaps between two consecutive prime numbers $p_r < p_{r+1} < 100$ occur successively when

$$\begin{aligned} p_{r+1} - p_r &= 5 - 3 = 2, \\ p_{r+1} - p_r &= 11 - 7 = 4, \\ p_{r+1} - p_r &= 29 - 23 = 6, \\ p_{r+1} - p_r &= 97 - 89 = 8. \end{aligned}$$

Is it true that these constantly increasing gaps always occur by jumps of length 2? In other words, does the first gap of length $2k$ always occur before the first gap of length $2k + 2$?

(207) Show that $\displaystyle\sum_{\alpha=2}^{\infty} \sum_{p} \frac{1}{p^\alpha} < 1$, where the inner sum runs over all the prime numbers p.

(208) Let

$$f(x) = \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \frac{1}{4}\pi(x^{1/4}) + \cdots,$$

be a series which is in fact a finite sum for each real number $x \geq 1$ since $\pi(x^{1/n}) = 0$ as soon as $n > \log x / \log 2$. Show that

$$\pi(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} f(x^{1/n}).$$

REMARK: It is possible to show that $f(x)$ is a better approximation of $\pi(x)$ than $Li(x) := \int_2^x \frac{dt}{\log t}$ (see H. Riesel [**31**]).

(209) Let $n \geq 2$ be an integer. Show that the interval $[n, 2n]$ contains at least one perfect square.

(210) If n is a positive integer such that $3n^2 - 3n + 1$ is composite, show that n^3 cannot be written as $n^3 = p + m^3$, with p prime and m a positive integer.

(211) It is conjectured that there exist infinitely many prime numbers p of the form $p = n^2 + 1$. Identify the primes $p < 10\,000$ of this particular form. Why is the last digit of such a prime number p always 1 or 7? Is there any reasonable explanation for the fact that the digit 7 appears essentially twice as often?

(212) Show that, for each integer $n \geq 2$,

$$(n!)^{1/n} \leq \prod_{p \leq n} p^{\frac{1}{p-1}}.$$

(213) For each integer $N \geq 1$, let $S_N = \{n^2 + 2 : 6 \leq n \leq 6N\}$. Show that no more than $\frac{1}{6}$ of the elements of S_N are primes.

(214) Let p be a prime number and consider the integer $N = 2 \cdot 3 \cdot 5 \cdots p$. Show that the $(p-1)$ consecutive integers

$$N+2, N+3, N+4, \ldots, N+p$$

are composite.

(215) Let $n > 1$ be an integer with at least 3 digits. Show that
  (a) $2|n$ if and only if the last digit of n is divisible by 2;
  (b) $2^2|n$ if and only if the number formed with the last two digits of n is divisible by 4;
  (c) $2^3|n$ if and only if the number formed with the last three digits of n is divisible by 8.

Can one generalize?

(216) For each integer $n \geq 2$, let

$$P(n) = \prod_{\substack{p|n \\ p > \log n}} \left(1 - \frac{1}{p}\right).$$

Show that $\lim_{n\to\infty} P(n) = 1$.

(217) Prove that there exists an interval of the form $[n^2, (n+1)^2]$ containing at least 1000 prime numbers.

(218) Use the Prime Number Theorem (see Theorem 17) in order to prove that the set of numbers of the form p/q (where p and q are primes) is dense in the set of positive real numbers.

(219) Show that the sum of the reciprocals of a finite number of distinct prime numbers cannot be an integer.

(220) Use the fact that there exists a positive constant c such that if $x \geq 100$,

$$(1) \qquad \sum_{p \leq x} \frac{1}{p} = \log\log x + c + R(x) \quad \text{with } |R(x)| < \frac{1}{\log x}$$

and moreover that, for $x \geq 2$,

$$\pi(x) := \sum_{p \leq x} 1 < \frac{3}{2} \frac{x}{\log x} \tag{2}$$

in order to prove that if $P(n)$ stands for the largest prime factor of n, then

$$\frac{1}{x} \#\{n \leq x : P(n) > \sqrt{x}\} = \log 2 + T(x) \quad \text{with } |T(x)| < \frac{9}{2} \frac{1}{\log x}. \tag{3}$$

Use this result to show that more than $\frac{2}{3}$ of the integers have their largest prime factor larger than their square root, or in other words that the density of the set of integers n such that $P(n) > \sqrt{n}$ is larger than $\frac{2}{3}$.

(221) Prove the following formula (due to Adrien-Marie Legendre (1752–1833)):

$$\pi(x) = \pi(\sqrt{x}) + \sum_{n | p_1 \cdots p_r} \mu(n) \left[\frac{x}{n} \right] - 1,$$

where $r = \pi(\sqrt{x})$.

(222) Consider the following two conjectures:

A. *(Goldbach Conjecture) Each even integer* ≥ 4 *can be written as the sum of two primes.*

B. *Each integer* > 5 *can be written as the sum of three prime numbers.*

Show that these two conjectures are equivalent.

(223) Show that $\pi(m)$, the number of prime numbers not exceeding the positive integer m, satisfies the relation

$$\pi(m) = \sum_{j=2}^{m} \left[\frac{(j-1)! + 1}{j} - \left[\frac{(j-1)!}{j} \right] \right],$$

where $[y]$ stands for the largest integer $\leq y$.

(224) Given a sequence of natural numbers $\mathcal{A}$, let $A(n) = \#\{m \leq n : m \in \mathcal{A}\}$, and let us denote respectively by

$$\underline{\mathbf{d}}\mathcal{A} = \liminf_{n \to \infty} \frac{A(n)}{n} \quad \text{and} \quad \overline{\mathbf{d}}\mathcal{A} = \limsup_{n \to \infty} \frac{A(n)}{n}$$

the *asymptotic lower density* and *asymptotic upper density* of the sequence $\mathcal{A}$. On the other hand, if both these densities are equal, we say that the sequence $\mathcal{A}$ has density $\mathbf{d}\mathcal{A} = \underline{\mathbf{d}}\mathcal{A} = \overline{\mathbf{d}}\mathcal{A}$. Prove that:

(a) the density of the sequence made up of all the multiples of a natural number a is equal to $1/a$;

(b) the density of the sequence made up of all the multiples of a natural number a which are not divisible by the natural number a_0 is equal to $\dfrac{1}{a} - \dfrac{1}{[a, a_0]}$;

(c) the density of the sequence made up of all natural numbers which are not divisible by any of the prime numbers $q_1, q_2, \ldots, q_r$ is equal to $\displaystyle\prod_{i=1}^{r} \left(1 - \frac{1}{q_i}\right)$.

(225) Let $\mathcal{A}$ be the set of natural numbers n such that $2^{2k} \leq n < 2^{2k+1}$ for a certain integer $k \geq 0$, so that

$$\mathcal{A} = \{1, 4, 5, 6, 7, 16, 17, \ldots, 31, 64, 65, \ldots, 127, 256, 257, \ldots\}.$$

Show that

$$\underline{\mathbf{d}}\,\mathcal{A} \neq \overline{\mathbf{d}}\,\mathcal{A}.$$

(226) We say that a sequence of natural numbers $\mathcal{A}$ is *primitive* if no element of $\mathcal{A}$ divides another one. Examples of such sequences are: the sequence of prime numbers, the sequence of natural numbers having exactly k prime factors (k fixed), and finally the sequence of integers n belonging to the interval $]k, 2k]$ (k fixed). Show that if $\mathcal{A}$ is a primitive sequence, then $\overline{\mathbf{d}}\,\mathcal{A} \leq \frac{1}{2}$.

(227) Let $\mathcal{A}$ be a primitive sequence (see Problem 226). Show that

$$\sum_{a\in\mathcal{A}} \frac{1}{a\log a} < +\infty.$$

(228) Let $E = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$.

(a) Show that the sum and the product of elements of E are in E.

(b) Define the norm of an element $z \in E$ by $\|z\| = \|a+b\sqrt{-5}\| = a^2+5b^2$. We say that an element $p \in E$ is *prime* if it is impossible to write $p = n_1n_2$, with $n_1, n_2 \in E$, $\|n_1\| > 1$, $\|n_2\| > 1$; we say that it is *composite* if it is not prime. Show that, in E, 3 is a prime number and 29 is a composite number.

(c) Show that the factorization of 9 in E is not unique.

(229) Let A be a set of natural numbers and let $A(x) = \#\{n \leq x : n \in A\}$. Show that, for all $x \geq 1$,

$$\sum_{\substack{n\leq x\\ n\in A}} \frac{1}{n} = \sum_{n\leq x} \frac{A(n)}{n(n+1)} + \frac{A(x)}{[x]+1}.$$

4. Representations of Numbers

(230) A number $n = d_1 d_2 \cdots d_r$, where $d_1, d_2, \ldots, d_r$ are the digits of n, is called a *palindrome* if it remains unchanged when its digits are reversed, that is if $n = d_r d_{r-1} \cdots d_1$. Hence the numbers 36763 and 437734 are both palindromes. Show that each palindrome having an even number of digits is divisible by 11.

(231) The smallest number $n > 2$ which is equal to the sum of the factorials of its digits in basis 15 is 1 441 (here, $1\,441 = [6, 6, 1]_{15} = 6! + 6! + 1!$). How can one find another such number $n > 2$ without using a computer?

(232) Let r be a positive integer and let n be a number which can be written as a sum of r distinct factorials, that is for which there exist positive integers $d_1 < d_2 < \ldots < d_r$ such that

$$n = d_1! + d_2! + \cdots + d_r!.$$

Prove that such a representation is unique.

(233) Let c be a positive odd integer. Show that the equation $x^2 - y^3 = 8c^3 - 1$ has no solutions in positive integers x and y, and use this to show that there exist infinitely many positive integers which are not of the form $x^2 - y^3$.

(234) Show that the last four digits of the decimal representation of 5^n, for $n = 4, 5, 6, \ldots$, form a periodic sequence. What is this period?

(235) Show that there exist infinitely many natural numbers which cannot be written as the sum of one, two or three cubes.

(236) Show that every integer can be written as the sum of five cubes.

(237) Given a positive integer n, let $s(n)$ be the sum of its digits, so that for example $s(12) = 3$ and $s(924) = 15$. Find all pairs of integers $m < n$ such that $s(m)^2 = n$ and $s(n)^2 = m$.

(238) The Egyptians used to express each fraction (except $\frac{2}{3}$) as a sum of unitary fractions (that is, fractions of the form $1/n$, where n is a positive integer).
   (a) Prove the result etablished by James Joseph Sylvester (1814–1897) to the effect that each fraction n/m, $n < m$, $(n, m) = 1$, can be written as a sum of unitary fractions.
   (b) Show that such a representation is not necessarily unique.
   (c) Show that if n is of the form $n = 4m + 3$, then $4/n$ is the sum of three unitary distinct fractions.

(239) A positive integer is said to be *complete* if its square uses each of the 10 digits exactly once. Use a computer to find the smallest complete number and then show that a complete number cannot be a prime number.

(240) Show that 2 is the only prime number p which can be written as $p = x^3 + y^3$ with $x, y \in \mathbb{N}$.

(241) Prove that a prime number p can be written as the difference of two positive cubes if and only if $p = 3k(k+1) + 1$ for a certain positive integer k. Find the ten smallest prime numbers of this form.

(242) Prove that an integer n can be written as the difference of two squares if and only if n is not of the form $4k + 2$.

(243) We say that an integer $n > 1$ is *automorphic* if the number n^2 ends with the same digits as n. Hence 5, 25 and 625 are automorphic numbers. Show that there exist infinitely many automorphic numbers.

(244) Show that for each number $n = d_1 d_2 \cdots d_r > 9$, where $d_1, d_2, \ldots, d_r$ stand for the digits of n in basis 10, we have

$$n - d_1 \cdot d_2 \cdots d_r \geq 10^{r-1}.$$

(245) Find the largest positive integer which is equal to the sum of the fifth powers of its digits added to the product of its digits.

(246) Show how one can construct the only sequence (a_k) of positive integers having the following properties:
(i) a_k is made of k digits;
(ii) 2^k divides a_k;
(iii) a_k contains only the digits 1 and 2.
Generate the first 14 terms of this sequence.

(247) For each positive integer n, let $s(n)$ be the sum of its digits. Given an integer $k \geq 2$ which is not a multiple of 3, let $\rho(k)$ be the smallest prime number p such that $s(p) = k$, if such a prime number p exists. In the particular case $k = 2$, it seems that there are only three prime numbers p such that $s(p) = 2$, namely 2, 11 and 101, and we have in particular that $\rho(2) = 2$. We also have that $\rho(4) = 13$, $\rho(5) = 5$, $\rho(7) = 7$, $\rho(8) = 17$, $\rho(10) = 19$, $\rho(11) = 29$, and so on; the function $\rho(k)$ increases quite fast; for instance, $\rho(80) = 998\,999\,999$. For each integer $k > 2$ which is not a multiple of 3, the candidates p such that $s(p) = k$ appear to be numerous, and in fact there seems to be infinitely many of them. It therefore appears that the function $\rho(k)$ is well defined. However, it is not at all obvious that given a particular integer $k > 2$, one can always find at least one prime number p such that $s(p) = k$. Nevertheless, prove that if $\rho(k)$ exists, then

$$\rho(k) \geq (a+1)10^b - 1, \quad \text{where } b = [k/9] \text{ and } a = k - 9b.$$

(248) Given a positive integer m, set

$$P(m) = \prod_{n \neq m} \frac{n^3 - m^3}{n^3 + m^3},$$

where the infinite product runs over all positive integers $n \neq m$. Show that

$$P(m) = (-1)^{m+1} \frac{2}{3} (m!)^2 \prod_{n=1}^{m} \frac{n+m}{n^3 + m^3}.$$

(249) Find all positive integers n which can be written as the sum of the factorials of their digits.

(250) Find the only positive integer $n = d_1 d_2 \cdots d_{2k}$, where $d_1, d_2, \ldots, d_{2k}$ stand for the digits (an even number of them) of n, such that

$$n = d_1^{d_2} \cdot d_3^{d_4} \cdots d_{2k-1}^{d_{2k}}.$$

More precisely, proceed in two steps. First, show that there exists only a finite number of positive integers with this property. Afterwards, use a computer to find this number, thereby elaborating a process which allows one to minimize the number of candidates.

(251) Show that there exist exactly six positive integers n with the property that the sum of their digits added to the sum of the cubes of their digits is equal to the number n itself, that is such that

$$n = d_1 d_2 \cdots d_r = d_1 + d_2 + \cdots + d_r + d_1^3 + d_2^3 + \cdots + d_r^3,$$

where $d_1, d_2, \ldots, d_r$ stand for the digits of n.

(252) Given a positive integer k, let $g(k)$ be the smallest number r which has the property that each positive integer can be written as $x_1^k + x_2^k + \cdots + x_r^k$, where the x_i's are nonnegative integers. In 1770, Joseph-Louis Lagrange (1736–1813) showed that $g(2) = 4$. The problem of calculating the value of $g(k)$ is known as the *Waring problem.* The mere fact that the function $g(k)$ is well defined is not at all obvious; as a matter of fact, it is only in 1909 that David Hilbert (1862–1943) finally showed that $g(k)$ exists for each positive integer k. It is conjectured that

$$g(k) = 2^k - 2 + \left[\left(\frac{3}{2} \right)^k \right] \qquad (k \geq 1).$$

Around 1772, Johannes Albert Euler (the son of the famous Leonhard Euler) proved that this last quantity is actually a lower bound for $g(k)$. Reconstruct this proof by considering the integer $n = q2^k - 1$, where the number q is defined implicitly by

$$3^k = q2^k + r \qquad (1 \leq r < 2^k).$$

(253) Find the only positive integer whose square and cube, taken together, use all the digits from 0 to 9 exactly once.

(254) Find the only positive integer whose square and cube, taken together, use all the digits from 0 to 9 exactly twice.

(255) Show that there are only a finite number of positive integers whose square and cube, taken together, uses all the digits from 0 to 9 exactly three times, and find these numbers.

(256) A positive integer n having $2r$ digits, $r \geq 1$, is called a *vampire number* if it can be written as the product of two positive integers, each of r digits, the union of their digits giving all the digits appearing in n. Hence $1260 = 21 \times 60$ is the smallest vampire number. Use a computer to find the seven vampire numbers made up of four digits.

(257) Given a positive integer $n = d_1 d_2 \cdots d_r$, where $d_1, d_2, \ldots, d_r$ stand for the digits of n, we let $g_3(n) = d_1^3 + d_2^3 + \cdots + d_r^3$. Find all positive integers n such that $g_3(g_3(n)) = n$.

(258) Given a positive integer $n = d_1 d_2 \cdots d_r$, where $d_1, d_2, \ldots, d_r$ stand for the digits of n, let $f(n) = d_1! + d_2! + \cdots + d_r!$. For each positive integer k, let f_k stand for the k-th iteration of the function f, that is $f_1(n) = f(n)$, $f_2(n) = f(f(n))$, and so on. Using a computer, show that, for every positive integer n, the iteration

$$(*) \qquad f_1(n), f_2(n), f_3(n), \ldots, f_k(n), \ldots$$

always ends up in an infinite loop. If $n = 1$, this loop is 1, 1, 1, ...; establish that if $n > 1$, then the iteration $(*)$ eventually enters one of the following six loops:

2, 2, 2, ...
145, 145, 145, ...
169, 363601, 1454, 169, ...
871, 45361, 871, ...
872, 45362, 872, ...
40585, 40585, 40585, ...

(259) Let n, k be arbitrary integers larger than 1. Show that there exists a polynomial $p(x)$ of degree k with integer coefficients and a positive integer m such that $n = p(m)$.

(260) If the number $111\ldots1$, made of k times the digit 1, is prime, show that k is prime.

(261) Prove that it is impossible to find three prime numbers $q_1 < q_2 < q_3$ such that

$$q_1 q_2 q_3 = q_1^3 + q_2^3 + q_3^3. \tag{1}$$

What if, instead of (1), we have

$$q_1 q_2 q_3 = q_1^2 + q_2^2 + q_3^2\,? \tag{2}$$

(262) Find all positive integers n such that $\frac{1}{n} = 0.\overline{n}$.

(263) Use a computer to find the eight positive composite integers $n < 10^6$ such that

$$n = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_r^{\alpha_r} = q_1^a + q_2 + \cdots + q_r, \tag{$*$}$$

for a certain positive integer a, where $q_1 < q_2 < \ldots < q_r$ are the prime factors of n.

(264) Show that $\sigma(n)$ is a power of 2 if and only if n is a product of Mersenne primes.

(265) What are the positive integers which can be represented in the form

$$\binom{k}{2} + kn, \qquad k > 1,\ n \geq 1?$$

5. Congruences

(266) For which positive integers n is the number $3^n + 1$ a multiple of 10?

(267) Find the smallest positive residue modulo 7 of $1! + 2! + \cdots + 50!$.

(268) What is the remainder of the division of $\sum_{i=1}^{111} i!$ by 12?

(269) Show that for each positive integer n, $10 \cdot 32^n + 1$ is a composite number.

(270) Is it true that 36 divides $n^6 + n^2 + 4$ for infinitely many positive integers n? Explain.

(271) In a letter sent to Christian Huygens (1629–1695) in 1659, Fermat wrote that using his method of infinite descent, he was successful in showing that no integer of the form $3k - 1$ can be written as $x^2 + 3y^2$ (with x and y integers). Is it possible to prove this result in a very simple manner? Explain.

(272) Let m and n be positive integers such that $p^m \| n$ for a certain prime number p. Show that

$$\frac{n!}{p^m} \equiv (-1)^m \prod_{k=0}^{\left[\frac{\log n}{\log p}\right]} \left(\left[\frac{n}{p^k}\right] - p\left[\frac{n}{p^{k+1}}\right]\right)! \pmod{p}.$$

(273) Let n be a positive integer. Show that the last digit of n^{13} is the same as the last digit of n.

(274) Find the smallest positive integer n such that $\sqrt[7]{n/7}$ and $\sqrt[11]{n/11}$ are both integers.

(275) Show that there exists an arbitrarily long sequence of consecutive integers, each divisible by a perfect square.

(276) Let a and b be integers and let m and n be positive integers. Show that the system of congruences

$$x \equiv a \pmod{m},$$
$$x \equiv b \pmod{n}$$

has solutions if and only if $(m, n) | (a - b)$.

(277) Let p be a prime number. Show that if k is an integer, $1 \le k < p$, then $\binom{p}{k} \equiv 0 \pmod{p}$.

(278) (a) Let $x_1, x_2, \ldots, x_n$ be integers. Show that

$$(x_1 + x_2 + \cdots + x_n)^p \equiv x_1^p + x_2^p + \cdots + x_n^p \pmod{p}.$$

(b) Show that if a and b are integers such that $a^p \equiv b^p \pmod{p}$, then $a^p \equiv b^p \pmod{p^2}$.

(279) Let p be an odd prime number and let k be an integer such that $1 \le k < p$. Show that

$$\binom{p-1}{k} \equiv (-1)^k \pmod{p}.$$

(280) Let p be a prime number and let r be an integer such that $1 \le r < p$. If $(-1)^r r! \equiv 1 \pmod{p}$, show that

$$(p - r - 1)! \equiv -1 \pmod{p}.$$

Use this result to show that $259! \equiv -1 \pmod{269}$ and $463! \equiv -1 \pmod{479}$.

(281) Let $\alpha \geq 3$ and $\beta \geq 6$ be two integers. Show that the equation $2^\beta - 1 = 3p^\alpha$ has no solutions for p prime.

(282) Let p be a prime number and let $n = 2p+1$. Show that if n is not a multiple of 3 and if $2^{n-1} \equiv 1 \pmod{n}$, then n is prime.

(283) Let p be a prime number and k a positive integer. Show that

$$(*) \qquad a \equiv b \pmod{p^k} \Longrightarrow a^p \equiv b^p \pmod{p^{k+1}}.$$

Then, prove that if $p > 2$, $p \not| a$ and $p^k \| a - b$, then $p^{k+1} \| a^p - b^p$.

(284) If p is a prime number, can the equation $p^\delta + 1 = 2^\nu$ have solutions with integers $\delta \geq 2$ and $\nu \geq 2$s?

(285) Show that the equation $1 + n + n^2 = m^2$, where m and n are positive integers, is impossible.

(286) Show that the only solution of the equation $1 + p + p^2 + p^3 + p^4 = q^2$, where p and q are primes, is $\{p, q\} = \{3, 11\}$.

(287) Let x_1, x_2, x_3, x_4 and x_5 be integers such that

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 0.$$

Show that necessarily one of the x_i's is a multiple of 7.

(288) Show that $2^p + 3^p$ is not a power (> 1) of an integer if p is prime.

(289) Show that for each positive integer n,

$$1^n + 2^n + 3^n + 4^n + 5^n + 6^n$$

is divisible by 7 if and only if n is not divisible by 6.

(290) Is it true that if n is a positive odd integer whose last digit in decimal representation is different from 5, then the last two digits of the decimal representation of n^{400} are 0 and 1? Explain.

(291) What are the possible values of the last digit of 4^m for each $m \in \mathbb{N}$?

(292) Show that the difference of two consecutive cubes is never divisible by 3, nor by 5.

(293) Is it true that $27 | (2^{5n+1} + 5^{n+2})$ for each integer $n \geq 0$? Explain.

(294) Show that for each positive integer k, the number $(13^2)^{2k+1} + (98^2)^{2k+1}$ is divisible by 337.

(295) Find the last two digits of the decimal representation of $19^{19^{19}}$.

(296) If a and b are positive integers such that $(ab, 70) = 1$, show that $a^{12} - b^{12} \equiv 0 \pmod{280}$.

(297) Show that for each integer $n \geq 2$, $n^{13} - n$ is divisible by 2730.

(298) Find the smallest positive integer which divided by 12, by 17, by 45 or by 70 gives in each case a remainder of 4.

(299) If n is an arbitrary positive integer, is the number

$$3n^{13} + 4n^{11} + n^7 + 3n^5 + 3n$$

divisible by 7?

(300) Let p be a prime number; show that $\binom{2p}{p} \equiv 2 \pmod{p}$.

(301) Show that a 3-digit positive integer whose decimal representation is of the form "abc" (for three digits a, b and c) is divisible by 7 if and only if $2a + 3b + c$ is divisible by 7.

(302) Show that a 6-digit positive integer whose decimal representation is of the form "*abcabc*" (for three digits a, b and c) is necessarily divisible by 13.

(303) Show that $561|2^{561}-2$ and that $561|3^{561}-3$.

(304) Given a positive integer n, show that

$$\frac{12}{35}n^{13}+\frac{23}{35}n$$

is an integer.

(305) Does there exist a rational number r such that for each positive integer n relatively prime with 481,

$$\frac{50}{481}n^{36}+r$$

is a positive integer?

(306) Let p be an odd prime number, $p\neq 5$. Show that p divides infinitely many integers amongst $1, 11, 111, 1\,111, \ldots$.

(307) According to Fermat's Little Theorem, if n is an odd prime number and if a is a positive integer such that $(a,n)=1$, then $a^{n-1}\equiv 1 \pmod{n}$. Show that the reverse of this result is false.

(308) Let $p>3$ be a prime number. Show that $ab^p-ba^p\equiv 0 \pmod{6p}$ for any integers a and b.

(309) If n is a positive integer, is it true that

$$1+2+3+\cdots+(n-1)\equiv 0 \pmod{n}?$$

Explain.

(310) For which positive integers n do we have

$$1^2+2^2+3^2+\cdots+(n-1)^2\equiv 0 \pmod{n}?$$

(311) Is it true that if n is a positive integer divisible by 4, then

$$1^3+2^3+3^3+\cdots+(n-1)^3\equiv 0 \pmod{n}?$$

(312) Prove that for each positive integer n, we have

$$5^n\equiv 1+4n \pmod{16} \quad \text{and} \quad 5^n\equiv 1+4n+8n(n-1) \pmod{64}.$$

(313) Show that for each positive integer $k\geq 3$,

$$5^{2^{k-3}}\not\equiv 1 \pmod{2^k} \quad \text{while} \quad 5^{2^{k-2}}\equiv 1 \pmod{2^k}.$$

More generally, show that for $k>2$ and a given odd integer a, we have

$$a^{2^{k-2}}\equiv 1 \pmod{2^k}.$$

(314) Show that

$$\frac{n^5}{5}+\frac{n^3}{3}+\frac{7n}{15}$$

is an integer for all $n\in\mathbb{N}$. More generally, show that if p and q are prime numbers, then

$$\frac{n^p}{p}+\frac{n^q}{q}+\frac{(pq-p-q)n}{pq}$$

is an integer for all $n\in\mathbb{N}$.

(315) Find the solution of the congruence $x^{24}+7x\equiv 2 \pmod{13}$.

(316) Because of Wilson's Theorem, the numbers $2,3,4\ldots,15$ can be arranged in seven pairs $\{x,y\}$ such that $xy\equiv 1 \pmod{17}$. Find these seven pairs.

(317) Let $m = m_1 m_2 \cdots m_r$, where the m_i's are integers > 1 and pairwise coprime. Show that

$$m_1^{\phi(m)/\phi(m_1)} + m_2^{\phi(m)/\phi(m_2)} + \cdots + m_r^{\phi(m)/\phi(m_r)} \equiv r - 1 \pmod{m}.$$

(318) Let p be a prime number and k an integer, $0 < k < p$. Show that

$$(k-1)!(p-k)! \equiv (-1)^k \pmod{p}.$$

(319) If p and q are distinct prime numbers, is it true that we always have

$$p^{q-1} + q^{p-1} \equiv 1 \pmod{pq}\,?$$

More generally, if m and n are positive integers such that $(m, n) = 1$, is it true that

$$n^{\phi(m)} + m^{\phi(n)} \equiv 1 \pmod{mn}\,?$$

(320) Show that for each positive integer n,

$$3^{2n+2} \equiv 8n + 9 \pmod{64}.$$

(321) Let $p \geq 5$ be a prime number. Find the value of $(p!, (p-2)! - 1)$.

(322) Show that

$$5^{6614} - 12^{857} \equiv 1 \pmod{7}.$$

(323) *Divisibility tests.* Let N be a positive integer whose decimal representation is $N = a_n 10^n + \cdots + a_2 10^2 + a_1 10 + a_0$, where $0 < a_n \leq 9$ and for $k = 0, \ldots, n-1$, $0 \leq a_k \leq 9$. Show that

(a) N is divisible by 3 $\iff a_n + a_{n-1} + \cdots + a_1 + a_0 \equiv 0 \pmod{3}$.

(b) N is divisible by 4 $\iff 10a_1 + a_0 \equiv 0 \pmod{4}$.

(c) N is divisible by 6 $\iff 4(a_n + \cdots + a_1 + a_0) \equiv 3a_0 \pmod{6}$.

(d) N is divisible by 7 $\iff (100a_2 + 10a_1 + a_0) - (100a_5 + 10a_4 + a_3) + (100a_8 + 10a_7 + a_6) - \cdots \equiv 0 \pmod{7}$.

(e) N is divisible by 8 $\iff 100a_2 + 10a_1 + a_0 \equiv 0 \pmod{8}$.

(f) N is divisible by 9 $\iff a_n + a_{n-1} + \cdots + a_0 \equiv 0 \pmod{9}$.

(g) N is divisible by 11 $\iff a_n - a_{n-1} + \cdots + (-1)^n a_0 \equiv 0 \pmod{11}$.

(324) Assume that 168 divides the integer whose decimal representation is "770ab45c". Find the digits a, b and c.

(325) Let a be an integer ≥ 2 and let $m \in \mathbb{N}$. If $(a, m) = (a-1, m) = 1$, show that

$$1 + a + a^2 + \cdots + a^{\phi(m)-1} \equiv 0 \pmod{m}.$$

(326) Let p be a prime number. Show that for each $a \in \mathbb{N}$, we have

$$a^{(p-1)!+1} \equiv a \pmod{p}.$$

(327) Show that if p is a prime number, then $1^{p-1} + 2^{p-1} + \cdots + (p-1)^{p-1} \equiv -1 \pmod{p}$.

(328) Show that if p is an odd prime number, then $1^p + 2^p + \cdots + (p-1)^p \equiv 0 \pmod{p}$.

(329) Let p be an odd prime number. Show that

$$\sum_{k=1}^{p-1} (k-1)!(p-k)!k^{p-1} \equiv 0 \pmod{p}.$$

(330) Letting p be a prime number of the form $4n+1$, show that $((2n)!)^2 \equiv -1 \pmod{p}$. More generally, if p is a prime number and if $m+n=p-1$, $m \geq 0$, $n \geq 0$, show that

$$m!\,n! \equiv (-1)^{m+1} \pmod{p}.$$

(A similar result was obtained in Problem 318.) Use this last formula to prove that

$$\left\{\left(\frac{p-1}{2}\right)!\right\}^2 \equiv (-1)^{(p+1)/2} \pmod{p}.$$

(331) Show that an integer $n > 2$ is prime if and only if n divides the number $2(n-3)!+1$.

(332) Show that if p is a prime number and a an arbitrary integer, then p divides the expression $a^p + a(p-1)!$.

(333) Show that if $\pi = 3.141592\ldots$ stands for Archimede's constant and $\pi(x)$ stands for the number of prime numbers $p \leq x$, then

$$\pi(x) = \sum_{2 \leq n \leq x} \left[\cos^2\left(\pi \frac{(n-1)!+1}{n}\right)\right],$$

where $[y]$ stands for the largest integer smaller or equal to y.

(334) Let $m_1, m_2 \in \mathbb{N}$ be such that $(m_1, m_2) = 1$. If a, r and s are positive integers such that $a^r \equiv 1 \pmod{m_1}$ and $a^s \equiv 1 \pmod{m_2}$. Show that

$$a^{[r,s]} \equiv 1 \pmod{m_1 m_2}.$$

(335) Let m be a positive integer. Show that for each $a \in \mathbb{N}$,

$$a^m \equiv a^{m-\phi(m)} \pmod{m}.$$

(336) Let m be a positive odd integer. Show that the sum of the elements of a complete residue system modulo m is congruent to $0 \pmod{m}$.

(337) Let $a, b \in \mathbb{Z}$, $m \in \mathbb{N}$. If E is a complete residue system modulo m and if $(a, m) = 1$, show that

$$E' = \{ax + b \mid x \in E\}$$

is also a complete residue system modulo m.

(338) Is it possible to construct a reduced residue system modulo 7 made up entirely of multiples of 6? Explain.

(339) Let $m > 2$ be an integer. Show that the sum of the elements of a reduced residue system modulo m is congruent to $0 \pmod{m}$.

(340) If $\{r_1, r_2, \ldots, r_{p-1}\}$ is a reduced residue system modulo a prime number p, show that

$$\prod_{j=1}^{p-1} r_j \equiv -1 \pmod{p}.$$

(341) Let $a, b \in \mathbb{Z}$, $m \in \mathbb{N}$. Using a counter-example, show that if E is a reduced residue system modulo m and if $(a, m) = 1$, then the set $\{ax + b \mid x \in E\}$ is not necessarily a reduced residue system modulo m.

(342) Find all integers x, y and z with $2 \leq x \leq y \leq z$ such that

$$xy \equiv 1 \pmod{z}, \quad xz \equiv 1 \pmod{y}, \quad yz \equiv 1 \pmod{x}.$$

(343) Let n and k be positive integers. Show that there exists a sequence of n consecutive composite integers such that each is divisible by at least k distinct prime numbers. Using this result, find the smallest sequence of four consecutive integers divisible by 3, 5, 7 and 11 respectively.

(344) Find all positive integers which give the remainder 1, 2 and 3 when divided respectively by 3, 4 and 5.

(345) Find the smallest integer $a > 2$ such that

$$2|a, \quad 3|a+1, \quad 4|a+2, \quad 5|a+3, \quad 6|a+4.$$

(346) Find the cycle and the period of $1/3$, $1/3^2$, $1/3^3$, $1/3^4$, $1/7$, $1/7^2$, $1/7^3$. Let p be an arbitrary prime number for which the period of $1/p$ is m. Using these computations, what should one conjecture regarding the periods of $1/p^2, 1/p^3, \ldots, 1/p^n$?

(347) The decimal expansion of $2/3 = 0.666\ldots$ consists in a repetition of $6 = 2 \cdot 3$. The same phenomenon occurs with the decimal expansion of $1/3 = 0.333\ldots$. Find all positive rational numbers a/b with $(a,b) = 1$, whose decimal expansion is formed by an infinite repetition of the product of its numerator and of its denominator.

(348) Show that the period of a fraction m/n with $m < n$, $(m,n) = 1$, $(n,10) = 1$ is the smallest positive integer h such that $10^h \equiv 1 \pmod{n}$.

(349) If m/n has the cycle $a_1 a_2 \cdots a_h$, show that $m | a_1 a_2 \cdots a_h$.

(350) If $m/n = 0.\overline{a_1 a_2 \ldots a_r}$, show that

$$\frac{m}{n} = \frac{a_1 a_2 \ldots a_r}{10^r - 1},$$

where the numerator is the number made up of the r digits $a_1, a_2, \ldots, a_r$ (and not of their product).

6. Primality Tests and Factorization Algorithms

(351) Let $d > 1$ be a proper divisor of the positive integer n. Prove that $2^{n-1} + 2^{d-1} - 1$ is a composite number.

(352) Prove that if a Mersenne number, that is a number of the form $2^q - 1$ where q is prime, is not squarefree, then it must be divisible by a Wieferich prime, that is a prime number p such that $2^{p-1} \equiv 1 \pmod{p^2}$.

(353) Find the three smallest prime factors of the number $n = 5^{96} - 7^{112}$.

(354) Let $m \geq 4$ be an even integer and let $a \geq 2$ be an integer. Show that $\dfrac{m^a}{2} + \dfrac{m}{2} - 1$ is a composite number.

(355) Show that the sequence $2^{2^n} + 3$, $n = 1, 2, \ldots$, contains infinitely many composite numbers.

(356) Use Problem 354 to prove that $2^{2^6} + 15$ is a composite number.

(357) Is it true that $2^{2^n} + 15$ is a prime number for each integer $n \geq 0$? If it is true, prove it. If it is false, provide a counter-example.

(358) By a close examination of the representation of the number n given in Problem 84, obtain that $973|n$ and therefore that 139 is a prime factor of n.

(359) Knowing that the number $n = 999\,951$ has a prime factor p such that $300 < p < 400$ and observing that $n + 49 = 10^6$, find this number p.

(360) Show that 127 is a prime divisor of $2^{21} - 1$.

(361) Find four prime factors of $2^{2^6} - 1$.

(362) Prove that at least one third of the integers of the form $n10^n + 1$ are composite.

(363) Use Problem 75 to show that 3, 7 and 31 are prime factors of $2^{30} - 1$ and that 31 and 127 are prime factors of $2^{35} - 1$.

(364) Let $n = 2^{30} - 1$. Show that $11|n$ without computing explicitly the value of n.

(365) Use Problem 75 to show that 2, 5, 7 and 13 are prime factors of $3^{12} - 1$ and that 2, 5, 7, 13, 41 and 73 are prime factors of $3^{24} - 1$.

(366) Given two integers a and m larger than 1, show that, if m is odd, then $a + 1$ is a divisor of $a^m + 1$. Use this result to obtain the factorization of 1001.

(367) Generalize the result of Problem 366 to obtain that if a and m are two integers larger than 1 and if $d \geq 1$ is an odd divisor of m, then $a^{m/d} + 1$ is a divisor of $a^m + 1$. Use this result to show that 101 is a factor of 1000001.

(368) Show that 7, 11 and 13 are factors of $10^{15} + 1$.

(369) Show that $n^4 + 4$ is a composite number for each integer $n \geq 2$. More generally show that if a is a positive integer such that $2a$ is a perfect square, then $n^4 + a^2$ is a composite number provided $n \geq \sqrt{2a}$.

(370) Show that there exist infinitely many composite numbers of the form $k10^k + 1$.

(371) Show that if the number $k + 2$ is prime, then it is a prime divisor of the number $2k^k + 1$.

(372) Find three factors of $2^{58} + 1$.

(373) Let $M_p = 2^p - 1$, where p is an odd prime number. Show that all the factors of M_p are of the form $2kp + 1$, where $k \in \mathbb{N}$.

(374) The primality test of Lucas-Lehmer may be read as follows: "Let p be an odd prime number. The Mersenne number $M_p = 2^p - 1$ is prime if and only if $M_p | S_{p-1}$, where $S_1 = 4$ and $S_{n+1} \equiv S_n^2 - 2 \pmod{M_p}$, $n \geq 1$." Use this test (and a computer) to prove that M_{61} is prime.

(375) Factor the number $n = 10^{48} - 1$. A computer may prove handy to obtain certain factors of n smaller than 10^9.

(376) In 1960, Waclaw Sierpinski (1882–1969) proved that there exist infinitely many integers k such that each of the numbers $N = k \cdot 2^n + 1$ ($n = 1, 2, 3, \ldots$) is composite. Three years later, Selfridge proved that the number $k = 78\,557$ is such a number. Prove this last result of Selfridge by establishing that, in this case, N is always divisible by 3, 5, 7, 13, 19, 37 or 73.

(377) Find three prime factors of $10^{27} + 1$.

(378) In order to obtain the factorization of the odd integer $n > 1$, it certainly helps to notice that, if n is composite, it is always possible to write n as

$$(*) \quad n = x^2 - y^2 = (x+y)(x-y) \quad \text{with } x, y \text{ positive integers, } x - y > 1,$$

thus revealing the factors $x + y$ and $x - y$ of n (see Problems 81 and 82). To obtain a representation of type $(*)$, we may proceed as follows. We look for an integer x such that $x^2 - n$ is a perfect square, that is such that

$$x^2 - n = y^2.$$

As a first value for x, we choose the smallest integer k such that $k^2 \geq n$, and then we try with $k+1$, and so on. By proceeding in this manner, it is clear that we will eventually find an integer x such that $x^2 - n$ is a perfect square, the reason being that n is odd and composite. This factorization method is called FERMAT'S FACTORIZATION METHOD.

To show the method, we take $n = 2001$. Since $\sqrt{n} = 44.7325\ldots$, we shall successively try several values of x starting with $x = 45$; we then obtain the following table:

x	$x^2 - n = ?$	Perfect square ?
45	$45^2 - 2001 = 24$	NO
46	$46^2 - 2001 = 115$	NO
47	$47^2 - 2001 = 208$	NO
48	$48^2 - 2001 = 303$	NO
49	$49^2 - 2001 = 400$	YES

Hence, $2001 = 49^2 - 20^2 = (49+20)(49-20) = 69 \cdot 29$, thus providing a factorization of 2001.

Proceed as above in order to factorize 2009, and then use Fermat's factorization method to find two proper divisors of $n = 289\,751$.

(379) Fermat's factorization method works very well when the odd integer n which is to be factored has two divisors of roughly the same size. But if $n = pq$, where $p < q$ are far apart, the number of steps to reach a factorization may be very large. But this difficulty may be overcome. For instance, take the number $n = 1\,254\,713$. Multiply this number by a small prime number p_0, the goal being to obtain a number $m = p_0 n = d_1 d_2$,

where d_1 and d_2 are two positive integers whose quotient d_2/d_1 is close to 1. Use this strategy to obtain the factorization of n.

(380) Assume that $n = pq$, where p and q are two prime numbers satisfying $p < q < 2p$. Let δ be the number defined by

$$\frac{q}{p} = 1 + \delta,$$

so that $0 < \delta < 1$. Show that the number of steps necessary to factorize n by using Fermat's factorization method is approximately $\dfrac{p\delta^2}{8}$.

(381) Knowing that the number $n = 188\,686\,013$ is the product of two prime numbers p and q such that

$$\left|\frac{p}{q} - 3\right| < \frac{1}{100},$$

find the factorization of n.

(382) Given an integer $r \geq 2$ and an odd integer $k \geq 5$, consider the number

$$n = r^k + r^{k-1} + \cdots + r^2 + r + 1.$$

Prove that the number n has at least three prime factors and moreover that they are distinct if $r \geq 3$ or if $r = 2$ and $k \geq 7$.

(383) Let k be a positive integer. Show that $\{2^k \pm 2^{k-1} \pm 2^{k-2} \pm \cdots \pm 2^1 \pm 1\}$ represents the set of all positive odd numbers $\leq 2^{k+1} - 1$.

(384) The number 11 is prime, while it is easy to check that the numbers 111, 1 111 and 11 111 are composite.

(i) Show that if a number of the form $\underbrace{111\ldots1}_{k} = (10^k - 1)/9$ is prime, then the number k is necessarily a prime.

(ii) Show that, if p is a prime number, then each prime factor of $(10^p - 1)/9$ is of the form $2jp + 1$ for a certain positive integer j.

(iii) Use a computer to find the five smallest prime numbers p such that the number corresponding to $(10^p - 1)/9$ is prime.

(iv) Use a computer to obtain the factorization of the numbers $(10^p - 1)/9$ for each prime number $p < 50$.

(385) Show that each positive integer n for which there exist positive integers k, x and y such that

$$(*) \qquad n = x^{2k+1} + y^{2k+1}$$

is composite.

(386) Let n be a positive odd integer for which there exists a prime number $p_0 < \sqrt{n}$ such that $p_0 \cdot n$ can be written as the sum of two positive cubes. Show that n must be a composite number.

(387) Consider the number $n = 52\,657\,403$. Show that $7n$ can be written as the sum of two cubes (one of which is rather small!) and conclude that n is composite and divisible by 719.

(388) Consider the number $n = 237\,749\,938\,896\,803$. Show that $11n$ can be written as the sum of two fifth powers (one of which is rather small!) and conclude that n is composite and divisible by 1213.

(389) Let $n \geq 3$ be a squarefree odd composite number. Show that if for each prime divisor p of n, we have $p - 1 | n - 1$, then n is a Carmichael number.

(390) Let $p \geq 5$ be a prime number such that $2p-1$ and $3p-2$ are primes. Show that the number $n = p(2p-1)(3p-2)$ is a Carmichael number.

(391) Use Korselt's Criterion (mentioned in the remark on the solution of Problem 389) in order to prove that each Carmichael number must have at least three distinct prime factors.

(392) In the remark attached to the solution of Problem 389, we observed that an integer $n = q_1 q_2 \cdots q_k$, where $k \geq 3$ and $2 < q_1 < q_2 < \ldots < q_k$ are prime numbers, is a Carmichael number if and only if

$$(*) \qquad q_j - 1 \Big| \prod_{i=1}^{k} q_i - 1 \qquad (j = 1, 2, \ldots, k).$$

Show that condition $(*)$ can be replaced by the condition

$$q_j - 1 \Big| \prod_{\substack{i=1 \\ i \neq j}}^{k} q_i - 1 \qquad (j = 1, 2, \ldots, k).$$

(393) Observing that

$$(*) \qquad 327\,763 = 30^3 + 67^3 = 51^3 + 58^3,$$

find the factorization of 327 763.

(394) Searching for a prime factor of $n = 48\,790\,373$, we observe that

$$7 \cdot n = 341\,532\,611 = 699^3 + 8^3.$$

Use this information to obtain the factorization of n.

(395) In 1956, Paul Erdős raised the question of obtaining the value of the smallest integer $n > 3$ such that $2^n - 7$ is prime. Use a computer to find this number n as well as the five next numbers n with the same property. Show that, in this search, one may ignore even integers n, the integers $n \equiv 1 \pmod 4$, the integers $n \equiv 7 \pmod{10}$ as well as the integers $n \equiv 11 \pmod{12}$.

(396) Let $a \geq 2$ be an integer and let p be a prime number such that p does not divide $a(a^2-1)$. Show that the number

$$n = \frac{a^{2p} - 1}{a^2 - 1}$$

is pseudoprime in basis a. Use this method to find pseudoprimes in basis 2 and 3.

(397) Show that there exist infinitely many pseudoprimes in basis 2.

(398) Let a and m be two positive integers such that $(a, m) = 1$. We say that s is the order of a modulo m if s is the smallest positive integer such that $a^s \equiv 1 \pmod m$. Show that if $a^n \equiv 1 \pmod m$, then $s | n$.

(399) (Lucas' Test) Let $n \geq 3$ be an integer such that for each prime factor q of $n-1$ there exists an integer $a > 1$ such that $a^{n-1} \equiv 1 \pmod n$ and $a^{(n-1)/q} \not\equiv 1 \pmod n$. Show that n is prime.

(400) Let $n = 10^{12} + 61$. First verify that $2^2 \cdot 5 \cdot 3947 \cdot 12667849$ is indeed the factorization of $n-1$, and then use Lucas' Test, explained in Problem 399 (with an appropriate choice of a), to show that n is prime.

(401) Use the primality test of Lucas, explained in Problem 399, to prove that the numbers $n = r^4 + 1$, where r takes successively the values 1910, 1916 and 1926, are all primes.

(402) Let $n = 10^{12} + 63$. Verify that $n - 1 = 2 \cdot 3^2 \cdot 7 \cdot 47 \cdot 168861871$, and then use Lucas' Test, explained in Problem 399 (with an appropriate choice of a), to show that n is prime.

(403) (Pollard $p - 1$ factorization method) Let n be a positive integer. Assume that n has an odd prime factor p such that $p-1$ has all its prime factors $\leq k$, where k is a relatively small positive integer (such as $k = 100$ or 1000 or 10000), so that $(p-1)|k!$. Let m be the residue modulo n of $2^{k!}$ and let $g = (m-1, n)$. Show that $g > 1$, thus identifying a factor of n.

(404) Use the Pollard $p - 1$ factorization method to find the smallest prime factor of the Fermat number $F_9 = 2^{2^9} + 1$.

(405) Use the Pollard $p - 1$ factorization method and a computer to factor the number 252123019542987435093029.

(406) Use the Pollard $p - 1$ factorization method and a computer to obtain the three prime factors of the Mersenne number

$$2^{71} - 1 = 2361183241434822606847.$$

(407) Use the Pollard $p - 1$ factorization method and a computer to factor the number 136258390321.

(408) Let $n = 302\,446\,877$. Let m be the quantity $2^{25!}$ modulo n. Show that $g = (m-1, n) = 17\,389$. Use the Pollard $p - 1$ factorization method to conclude that 17 389 is a (prime) divisor of 302 446 877.

(409) Show that each prime factor p of the Fermat number $F_n = 2^{2^n} + 1$ with $n \geq 2$ is of the form $p = k \cdot 2^{n+2} + 1$, $k \in \mathbb{N}$.

(410) Use the result of Problem 409 in order to prove that 641 is a prime factor of $F_5 = 2^{2^5} + 1 = 4\,294\,967\,297$.

(411) Use the result of Problem 409 in order to prove that 274 177 is a prime factor of

$$F_6 = 2^{2^6} + 1 = 18\,446\,744\,073\,709\,551\,617.$$

(412) (Pepin's Test) Let $F_n = 2^{2^n} + 1$ be a Fermat number and let $k > 2$ be an integer. Show that, for $n \geq 2$,

$$F_n \text{ is prime and } \left(\frac{k}{F_n}\right) = -1 \qquad \Longleftrightarrow \qquad k^{\frac{F_k - 1}{2}} \equiv -1 \pmod{F_n}.$$

7. Integer Parts

(413) Let $\alpha, \beta \in \mathbb{R}$. Show that

(a) $[\alpha] + [\beta] + [\alpha + \beta] \le [2\alpha] + [2\beta]$;

(b) $[\alpha] + [\beta] + 2[\alpha + \beta] \le [3\alpha] + [3\beta]$;

(c) $[\alpha] + [\beta] + 3[\alpha + \beta] \le [4\alpha] + [4\beta]$;

(d) $2[\alpha] + 2[\beta] + 2[\alpha + \beta] \le [4\alpha] + [4\beta]$;

(e) $3[\alpha] + 3[\beta] + [\alpha + \beta] \le [4\alpha] + [4\beta]$.

(414) Show that $\dfrac{(2n)!}{(n!)^2}$ is an even integer for each $n \in \mathbb{N}$.

(415) Let $m, n \in \mathbb{N}$. Show that:

(a) $\dfrac{(2m)!(2n)!}{m!n!(m+n)!}$ is an integer; (b) $\dfrac{(4m)!(4n)!}{n!m!\big((m+n)!\big)^3}$ is an integer.

(416) Let $a_i \ge 0$, $i = 1, 2, \ldots, r$, be integers such that $a_1 + a_2 + \cdots + a_r = n$. Show that $\dfrac{n!}{a_1!a_2!\cdots a_r!}$ is an integer.

(417) How many zeros appear at the end of the decimal representation of $23!$?

(418) Show that the last digit of $n!$ which is different from 0 is always an even number provided $n \ge 5$.

(419) Find all positive integers n for which the number of zeros appearing at the end of the decimal representation of $n!$ is 57. What happens when the number of zeros is 60 or 61?

(420) Let n be a positive integer.

(a) Show that the largest integer α such that 5^α divides $(5^n - 3)!$ is $\dfrac{5^n - 4n - 1}{4}$.

(b) Let p be a prime number and i a positive integer smaller than p. Show that the largest integer α such that p^α divides $(p^n - i)!$ is

$$\frac{p^n - (p-1)n - 1}{p-1}.$$

(421) Let n be a positive integer. Find a formula which reveals explicitly, for a given prime number p, the unique value of α such that

$$p^\alpha \parallel \prod_{i=1}^{n} (2i).$$

(422) Let n be a positive integer. Find a formula which reveals explicitly, for a given prime number p, the unique value of α such that

$$p^\alpha \parallel \prod_{i=0}^{n} (2i+1)$$

and use this to show that

$$n = \sum_{k=1}^{\infty} \left(\left[\frac{2n+1}{2^k} \right] - \left[\frac{n}{2^k} \right] \right).$$

(423) Find all natural numbers n having the property that $[\sqrt{n}\,]$ is a divisor of n.

(424) Prove that for each integer $n \geq 1$,

$$\left[\sqrt{n}+\sqrt{n+1}\right] = \left[\sqrt{4n+1}\right] = \left[\sqrt{4n+2}\right] = \left[\sqrt{4n+3}\right].$$

(425) Prove that for each integer $n \geq 0$,

$$\left[\sqrt{n}+\sqrt{n+1}+\sqrt{n+2}\right] = \left[\sqrt{9n+8}\right].$$

(426) Let m and k be positive integers. Show that

$$\left[\frac{m-k}{k}\right] + \left[-\left(\frac{m+1}{k}\right)\right] + 2 = 0.$$

(427) Show that

$$\lim_{m\to\infty} [\cos^2(m!\pi x)] = \begin{cases} 0 & \text{if } x \in \mathbb{R}\setminus\mathbb{Q}, \\ 1 & \text{if } x \in \mathbb{Q}, \end{cases}$$

where $[y]$ stands for the largest integer smaller or equal to y, and thus establish that the function $f : \mathbb{R} \to \{0,1\}$ defined by

$$f(x) = \lim_{m\to\infty} [\cos^2(m!\pi x)]$$

represents the characteristic function of the rational numbers.

(428) Show that, for each positive integer n,

$$\left[\frac{n+1}{2}\right] + \left[\frac{n+2}{4}\right] + \left[\frac{n+4}{8}\right] + \left[\frac{n+8}{16}\right] + \cdots = n.$$

(429) Show that for each $m \in \mathbb{Z}$,

$$\left[\frac{m - \left[\frac{m-17}{25}\right]}{3}\right] = \left[\frac{8m+13}{25}\right].$$

(430) Let $m \in \mathbb{Z}$. Show that the expression

$$\left[\frac{3m+4}{13}\right] - \left[\frac{m - 28 - \left[\frac{m-7}{13}\right]}{4}\right]$$

does not depend on m.

(431) Given an integer $n \geq 2$, show that, for each positive integer $k < n$,

$$(*) \qquad \sum_{j=1}^{k} \left[\frac{n-j}{k}\right] = n-k.$$

(432) Show that if $\alpha \in \mathbb{R}$, we have $[\alpha] + \left[\alpha + \frac{1}{n}\right] + \cdots + \left[\alpha + \frac{n-1}{n}\right] = [n\alpha]$ for each positive integer n.

(433) Show that if $\alpha \in \mathbb{R}$, we have $\left[\frac{\alpha}{n}\right] + \left[\frac{\alpha+1}{n}\right] + \cdots + \left[\frac{\alpha+n-1}{n}\right] = [\alpha]$ for each positive integer n.

(434) Let m and n be positive integers such that $(m,n)=1$. Show that

$$\sum_{k=1}^{n-1}\left[\frac{mk}{n}\right]=\frac{(m-1)(n-1)}{2}.$$

(435) Let m and n be positive integers such that $(m,n)=d$. Show that

$$\sum_{k=1}^{n-1}\left[\frac{mk}{n}\right]=\frac{(m-1)(n-1)}{2}+\frac{d-1}{2}.$$

(436) Establish the formula obtained in 1997 by Marcelo Polezzi that provides the value of the greatest common divisor of two positive integers m and n:

$$(m,n)=2\sum_{j=1}^{m-1}\left[\frac{jn}{m}\right]+m+n-mn.$$

(437) Consider the arithmetical function f defined by

$$f(n)=(n+1)^2+n-\left[\sqrt{(n+1)^2+n+1}\right]^2 \qquad (n=1,2,3,\ldots).$$

Evaluate the quantity $f(n)-n$.

(438) Given an integer $n\geq 2$, show that

$$n!=\prod_{p\leq n}p^{\alpha_p}, \qquad \text{where} \qquad \alpha_p=\frac{n-s_p(n)}{p-1},$$

where $s_p(n)$ stands for the sum of the digits of n in basis p.

(439) Evaluate the series

$$\sum_{n=1}^{\infty}\frac{2^{\|\sqrt{n}\|}+2^{-\|\sqrt{n}\|}}{2^n},$$

where $\|x\|$ stands for the closest integer to x.

(440) The characteristic function of the odd numbers defined, for each integer $n\geq 1$, by

$$\chi(n)=\begin{cases}1 & \text{if } n \text{ is odd,}\\ 0 & \text{if } n \text{ is even,}\end{cases}$$

can be written in a single expression by using the function $[x]$; indeed,

$$\chi(n)=n-2\left[\frac{n}{2}\right] \qquad (n\geq 1).$$

Find a similar simple formula for the function

$$f(n)=\begin{cases}1 & \text{if } n \text{ is odd,}\\ 2 & \text{if } n \text{ is even.}\end{cases}$$

(441) In Problem 1, we established the two formulas

$$1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6} \qquad (n=1,2,3,\ldots),$$

$$1^3+2^3+3^3+\cdots+n^3=\frac{n^2(n+1)^2}{4} \qquad (n=1,2,3,\cdots).$$

Establish similar formulas for the two sums

$$A_n = [1^{1/2}] + [2^{1/2}] + [3^{1/2}] + \cdots + [(n^2-1)^{1/2}] \qquad (n = 2, 3, 4, \ldots),$$

$$B_n = [1^{1/3}] + [2^{1/3}] + [3^{1/3}] + \cdots + [(n^3-1)^{1/3}] \qquad (n = 2, 3, 4, \ldots),$$

where, as usual, $[x]$ stands for the largest integer $\leq x$.

(442) Let a be the positive solution of the quadratic equation $x^2 - x - 1 = 0$. Show that for each $n \in \mathbb{N}$, we have

$$[a^2 n] = [a[an] + 1].$$

(443) Show that for each $n \in \mathbb{N}$, the positive solution a of the equation $x^2 - x - 1 = 0$ verifies the equation

$$2[a^3 n] = [a^2[2an] + 1].$$

(444) Show that the number N of positive integer solutions x, y of the inequality $xy \leq n$, where n is a fixed positive integer, is given by

$$N = \left[\frac{n}{1}\right] + \left[\frac{n}{2}\right] + \cdots + \left[\frac{n}{n}\right] = 2\sum_{k=1}^{[\sqrt{n}]} \left[\frac{n}{k}\right] - [\sqrt{n}]^2.$$

(445) Let $n \in \mathbb{N}$. For each integer $k \geq 0$, find the number of integers i $(1 \leq i \leq n)$ which are divisible by 2^k but not by 2^{k+1}. Establish also that

$$\sum_{j=1}^{\infty} \left[\frac{n}{2^j} + \frac{1}{2}\right] = n.$$

Thus, by doing so, it will have been proved that one can evaluate the sum $\frac{n}{2} + \frac{n}{4} + \frac{n}{8} + \cdots$ by substituting each term by its closest integer (choosing the largest one if two such numbers exist).

(446) Show that for each $\alpha \in \mathbb{R}$, we have $\displaystyle\lim_{n\to\infty} \frac{[n\alpha]}{n} = \alpha$.

(447) Show that for each nonnegative real number α and for each positive integer k, we have $[\alpha^{1/k}] = [[\alpha]^{1/k}]$.

8. Arithmetical Functions

(448) One of the nice properties of Euler's ϕ function is given by the formula $\sum_{d|n} \phi(d) = n$. Using computer software, write a program which allows one to compute the value of $\sum_{d|n} f(d)$, where f is a given arithmetical function, for different values of n, for example, for $n = 1, \ldots, 100$.

(449) Use the program called for in Problem 448 to show that, for $n = 1, \ldots, 1000$,

$$\sum_{d|n} \tau(d) \equiv 0 \pmod 3$$

except for $n = k^3$, $k \in \mathbb{N}$. To do so, write a program that confirms that indeed this is true for $n = 1, \ldots, 1000$.

(450) Is it true that, for each integer $n \geq 0$, there exists a perfect number located between $10^n + 1$ and $10^{n+1} + 1$? Is it true that the last digit of the perfect numbers alternates between 6 and 8? Here, the use of a computer may prove useful.

(451) Show that if f and g are multiplicative functions, then their product fg is also a multiplicative function. If f is a multiplicative function, can one say that kf, for $k \in \mathbb{R}$, is also a multiplicative function? What about $f + g$?

(452) Does there exist a multiplicative function f such that

$$f(30) = 0, \quad f(105) = 1 \qquad \text{and} \qquad f(70) = 1?$$

(453) Let $t_1 = 1, t_2 = 3, t_3 = 6, \ldots, t_k = k(k+1)/2, \ldots$ be the sequence of triangular numbers. Let f be the arithmetical function defined by $f(n) = 1/k$, where k is the only integer satisfying $t_{k-1} < n \leq t_k$, so that

$$\sum_{n \leq t_k} f(n) = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \cdots + \underbrace{\frac{1}{k} + \cdots + \frac{1}{k}}_{k} = k,$$

for each positive integer k. Prove that

$$f(n) = \left\| \frac{1}{2}\sqrt{8n - 7} \right\|^{-1},$$

where $\|x\|$ stands for the closest integer to x.

(454) Let f be the arithmetical function defined by

$$f(n) = [\sqrt{n} - 1] + [\sqrt[3]{n} - 1] + [\sqrt[4]{n} - 1] + \cdots \qquad (n = 1, 2, \ldots).$$

Show that $\limsup_{n\to\infty}(f(n) - f(n-1)) = +\infty$.

(455) Indicate which amongst the following functions are totally multiplicative:

$$\gamma(n) = \prod_{p|n} p, \qquad \sigma_2(n) = \sum_{d|n} d^2,$$

$$g(n) = 2^{\omega(n)}, \qquad h(n) = \sum_{d|n} \mu^2(d)d, \qquad \rho(n) = 2^{\Omega(n)}.$$

(456) Show that the function $f(n) = [\sqrt{n}] - [\sqrt{n-1}]$ is multiplicative. Is this function totally multiplicative?

(457) A function f is said to be *strongly multiplicative* if it is multiplicative and if also $f(p^\alpha) = f(p)$ for each prime number p and each $\alpha \in \mathbb{N}$. Identify those functions, amongst the ones given below, that are strongly multiplicative:

$$\gamma(n) = \prod_{p|n} p, \quad \sigma_2(n) = \sum_{d|n} d^2, \qquad g(n) = 2^{\omega(n)}, \quad h(n) = \sum_{d|n} \mu^2(d)d.$$

(458) Let f be a multiplicative function. Is the function g defined by $g(n) = \sum_{d|n} \mu^2(d)f(d)$ necessarily strongly multiplicative?

(459) Let f be an arithmetical function which is both strongly multiplicative and totally multiplicative. Is it true that $\{f(n) : n = 1, 2, 3, \ldots\}$ contains at most two elements? Explain.

(460) Let g be an arithmetical function defined, for each $n \geq 1$, by

$$g(n) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod 3, \\ 2 & \text{if } n \equiv 1 \pmod 3, \\ 3 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Is it true that g is a multiplicative function? Explain.

(461) Prove that an arithmetical function f such that $f(1) = 1$ is multiplicative if and only if, for each $m, n \in \mathbb{N}$,

$$f((m, n))f([m, n]) = f(m)f(n).$$

(462) Let $\gamma(n)$ be the arithmetical function which represents the kernel of n, that is $\gamma(n) = \prod_{p|n} p$. Show that:
(a) γ is a multiplicative function;
(b) $\gamma(n) = \sum_{d|n} |\mu(d)|\phi(d)$ for each integer $n \geq 1$.

(463) Show that if the *abc* conjecture (see page 12) is true, then for all $\varepsilon > 0$, there exists a constant $M = M(\varepsilon)$ such that for each integer $n \geq 2$, we have

$$n < M \cdot \gamma(n^2 - 1)^{1+\varepsilon},$$

where $\gamma(m)$ is the product of the prime factors of m.

(464) Let f be a multiplicative function and let k be a positive integer such that $f(k) \neq 0$. Show that the arithmetical function g defined by

$$g(n) := \frac{f(kn)}{f(k)} \quad \text{is also multiplicative.}$$

(465) Let $f : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function such that $f(2) = 2$ and $f(mn) = f(m)f(n)$ if m and n are relatively prime. Show that f is the identity function, that is that $f(n) = n$ for each $n \geq 1$.

(466) Let f be an additive function. Assume that, for each positive integer n, $\displaystyle\lim_{k\to\infty} \frac{f(n^k)}{k}$ exists. Show that the function g defined by $g(n) = \displaystyle\lim_{k\to\infty} \frac{f(n^k)}{k}$ is totally additive.

(467) Let f and g be two multiplicative functions. Show that the function h defined by

$$h(n) = \sum_{\substack{dr=n \\ (d,r)=1}} f(d)g(r) \qquad (n = 1, 2, \ldots)$$

is also a multiplicative function.

(468) Let f and g be two multiplicative functions. Show that the function h defined by

$$h(n) = \sum_{[d,r]=n} f(d)g(r) \qquad (n = 1, 2, \ldots),$$

where the sum runs over all ordered pairs (d, r) such that $[d, r] = n$ is a multiplicative function.

(469) Let f be a multiplicative function such that, for each prime number p, $\lim_{k\to\infty} f(p^k) = 0$. Do we necessarily have that $\lim_{n\to\infty} f(n) = 0$? Explain.

(470) Let f be a multiplicative function such that, for each positive integer k, $\lim_{p\to\infty} f(p^k) = 0$. Do we necessarily have that $\lim_{n\to\infty} f(n) = 0$? Explain.

(471) Let f be a totally additive function which is monotonically increasing; prove that there exists a constant $c \geq 0$ such that $f(n) = c \log n$ for each integer $n \geq 1$.

(472) Consider the arithmetical function f defined by $f(1) = 1$ and for $n > 1$ by

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ 3n + 1 & \text{if } n \text{ is odd.} \end{cases}$$

Define the functions f^0, f^1, $f^2, \ldots$ as follows: $f^0(n) = n$, $f^1(n) = f(n)$, $f^2(n) = f(f(n))$, $f^3(n) = f(f^2(n)), \ldots$, $f^k(n) = f(f^{k-1}(n)), \ldots$. The *Collatz Problem* (also called the *Syracuse Problem*) consists of attempting to establish that for each positive integer n, there exists $k \in \mathbb{N}$ such that $f^k(n) = 1$. This result is most likely true, but no one has ever been able to prove it. However, partial results have been obtained.

(a) Let α and j be two positive integers. What is the value of $f^j(2^\alpha)$?
(b) Let $\alpha \in \mathbb{N}$. For which values of $j \in \mathbb{N}$ is it true that $f^j(2^\alpha) = 1$?
(c) What is the smallest value of $n \in \mathbb{N}$ such that $f^k(n) = 11$ for a certain positive integer k?
(d) Show that, if n is a positive odd integer, then $f^3(n) < \frac{3}{4}n + 1$ if $n \equiv 1 \pmod 4$ while $f^3(n) > 4n$ if $n \equiv 3 \pmod 4$.
(e) Find an integer n such that $f^{2k+1}(n)$ is odd for $k = 0, 1, 2, 3, 4$ and such that $f^{2k}(n)$ is even for $k = 0, 1, 2, 3, 4, 5$.
(f) Is it true that if $f^3(n) > n$, then $f^3(n + 2) < n + 2$?
(g) Given an odd positive odd n, what is the probability that $f^3(n)$ is larger than n?
(h) Consider the arithmetical function g defined by $g(1) = 1$ and for $n > 1$ by

$$g(n) = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ 5n + 1 & \text{if } n \text{ is odd,} \end{cases}$$

and then define the functions g^0, g^1, $g^2, \ldots$ as we did for the function f. Show that the conjecture to the effect that "for each $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $g^k(n) = 1$" is false.
(i) Let $n \geq 3$. We introduce the function $\mathsf{Syr}(n)$ which stands for the smallest positive integer α such that $f^\alpha(n) = 1$. For instance,

$\mathsf{Syr}(8) = 3$. Show that $\mathsf{Syr}(n) \geq \log_2 n$, where $\log_2(n)$ stands for the logarithm of n in basis 2.

(j) Let $n \geq 3$ and let $\mathsf{Syr}(n)$ be the function introduced above. Prove that if n is odd, then $\mathsf{Syr}(n) \geq \log_2 n + 3$.

(k) Let $\alpha \in \mathbb{N}$ and consider the number $n = 2^{\alpha+1} - 1$. Show that

$$f^{2k}(n) = 3^k \cdot 2^{\alpha-k+1} - 1 \qquad \text{for each integer } k,\ 1 \leq k \leq \alpha,$$

and therefore that the sequence of iterations

$$f^2(n), f^4(n), f^6(n), \ldots, f^{2\alpha}(n)$$

is strictly increasing.

(l) Let α and n be as in (k). Show that

$$f^{2\alpha}(n) = 2 \cdot 3^\alpha - 1.$$

(m) Let $\alpha \in \mathbb{N}$ and consider the sequence of integers $n_0, n_1, n_2, \ldots$ defined by $n_j = j2^\alpha + (2^\alpha - 1)$. Show that

$$f^{2\alpha}(n_j) = (j+1)3^\alpha - 1, \qquad \text{for each } j.$$

(n) Given an arbitrary large real number $C > 0$, show that there exist two positive integers n and k such that $f^k(n) > Cn$.

(o) Consider the arithmetical function f_* defined by $f_*(1) = 1$ and for $n > 1$ by

$$f_*(n) = \begin{cases} n/2^\beta & \text{if } n = 2^\beta r, \text{ with } \beta \geq 1 \text{ and } r \text{ odd}, \\ 3n+1 & \text{if } n \text{ is odd}, \end{cases}$$

and then define the iteration functions f_*^0, f_*^1, f_*^2, ... as we did for the function f, and establish a table of the values $f_*^0(n), f_*^1(n), f_*^2(n), \ldots,$ $f_*^{10}(n)$ for $n = 1, 2, 3, \ldots, 50$. Now, given an arbitrary positive integer n, can we conclude that there necessarily exists a positive integer k such that $f_*^k(n) = 1$? Is there here an analogy with the "standard" Collatz Problem?

(473) Let a, b, c be positive integers such that $(a, b, c) = 1$. Is it true that

$$\tau(abc) = \tau(a)\tau(b)\tau(c)\,?$$

(474) Find the smallest positive integer n such that
(a) $\tau(n) = 9$; (b) $\tau(n) = 10$; (c) $\tau(n) = 15$.

(475) Identify all natural numbers having exactly 14 divisors.

(476) Find the largest prime number p such that
(a) $p|\tau(20!)$; (b) $p|\sigma(20!)$; (c) $p^2|\tau(35!)$; (d) $p^2|\sigma(35!)$.

(477) How many positive integers n are there dividing at least one of the two numbers 10^{40} and 20^{30}?

(478) Prove that

$$\sum_{n=1}^{\infty} \frac{\tau(n)}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2^n - 1}.$$

(479) Consider the sequence (b_k) defined by

$$b_1 = 2, \qquad b_{k+1} = 1 + b_1 b_2 \cdots b_k \quad (k = 1, 2, \ldots).$$

Show that
(i) for each integer $k \geq 1$, $b_{k+1} = b_k^2 - b_k + 1$,

(ii) $\displaystyle\sum_{j=1}^{\infty} \frac{1}{b_j} = 1$.

Use this to show that the arithmetical function g defined by

$$g(n) = n - 1 - \sum_{j=1}^{\infty} \left[\frac{n-1}{b_j} \right]$$

has the representation

$$g(n) = \sum_{j=1}^{\infty} \left\{ \frac{n-1}{b_j} \right\},$$

where $\{x\}$ stands for the fractional part of x.

(480) Let $\tau_1(n)$ be the number of odd divisors of n. Prove that τ_1 is a multiplicative function.

(481) Given a positive integer n, show that the number of ordered pairs of positive integers a, b such that $ab = n$ and $(a, b) = 1$ is $2^{\omega(n)}$.

(482) Let n be a positive integer. Show that the number of ordered pairs of positive integers a, b such that $[a, b] = n$ is $\tau(n^2)$.

(483) Let d and n be positive integers such that $d^2|n$. Show that the number of ordered pairs of positive integers a, b such that $(a, b) = d$ and $ab = n$ is $2^{\omega(n/d^2)}$. Use this to show that

$$\tau(n) = \sum_{d^2|n} \omega(n/d^2).$$

(484) Let n be a positive integer and let 2^α be the largest power of 2 that divides n. To which of the following five values is the quotient $\tau(2n)/\tau(n)$ equal:

$$2, \quad \frac{\alpha+3}{\alpha+2}, \quad \frac{\alpha+2}{\alpha+1}, \quad \frac{\alpha+1}{\alpha}, \quad \frac{\alpha}{\alpha-1} \ ?$$

Explain.

(485) Show that $\tau(n)$ is odd if and only if n is a perfect square.

(486) Show that if $\sigma(n)$ is a prime number for a certain positive integer n, then $\tau(n)$ must also be a prime number.

(487) Show that $\sigma(n)$ is odd if and only if n is a square or two times a square.

(488) Show that, for each positive integer n,

$$\prod_{d|n} d = n^{\tau(n)/2}.$$

What happens if $\tau(n)$ is odd?

(489) Prove that for each integer $n \geq 1$, we have $\sigma_2(n) \geq n\tau(n)$, where $\sigma_2(n) = \sum_{d|n} d^2$.

(490) Find the minimal value of $\tau(n(n+1))$ as n runs through the positive integers greater or equal to 3.

(491) For each integer $n \geq 1$, consider the functions $f_1(n)$ and $f_2(n)$ which stand respectively for the product of the odd divisors of n and for the product of the even divisors of n. Establish the following formulas:

$$\begin{aligned} f_1(n) &= m^{\tau(m)/2}, \\ f_2(n) &= \left(2^{\alpha(\alpha+1)} m^\alpha\right)^{\tau(m)/2} = (2n)^{\alpha\tau(m)/2}, \end{aligned}$$

where m and α are defined implicitly by $n = 2^\alpha m$, m odd.

(492) Show that $\prod_{m=1}^{n} m^{2[n/m]-\tau(m)} = 1$, where $[y]$ stands for the largest integer $\le y$.

(493) Given a positive integer n, consider the corresponding sequence

$$n,\ \tau(n),\ \tau(\tau(n)),\ \tau(\tau(\tau(n))),\ldots.$$

Identify those positive integers n for which the above sequence contains no perfect squares.

(494) For each real number a, define the function σ_a by $\sigma_a(n) = \sum_{d|n} d^a$. It is clear that $\tau(n)$ and $\sigma(n)$ are particular cases of $\sigma_a(n)$. Prove that

$$\sigma_a(n) = \begin{cases} \displaystyle\prod_{p^\alpha \| n} \frac{p^{a(\alpha+1)}-1}{p^a-1} & \text{if } a \neq 0, \\ \displaystyle\prod_{p^\alpha \| n} (\alpha+1) & \text{if } a = 0. \end{cases}$$

(495) Assume that p and q are odd prime numbers, and a and b are positive integers such that $p^a > q^b$. Show that, if p^a divides $\sigma(p^a)\sigma(q^b)$, then $p^a = \sigma(q^b)$.

(496) Let $\sigma^*(n)$ be the sum of the odd divisors of n. Show that σ^* is a multiplicative function.

(497) Show that $3|\sigma(3n-1)$ and $4|\sigma(4n-1)$ for each positive integer n. Is it true that $12|\sigma(12n-1)$ for each $n \ge 1$? Explain.

(498) Let p be a prime number and let a and b be nonnegative integers. Show that $\sigma(p^a)|\sigma(p^b)$ if and only if $(a+1)|(b+1)$.

(499) Show that $\sigma_{-a}(n) = n^{-a}\sigma_a(n)$ for each real number a and each positive integer n. In particular, show that the sum of the reciprocals of the divisors of a positive integer n is equal to $\sigma(n)/n$.

(500) Show that n is an even perfect number if and only if there exists a positive integer k such that $n = 2^{k-1}(2^k-1)$, where $2^k - 1$ is a prime number.

(501) In 1958, Perisatri proved that if n is an odd perfect number, then

$$\frac{1}{2} < \sum_{p|n} \frac{1}{p} < 2\log\frac{\pi}{2}.$$

Is it true that these inequalities still hold for each even perfect number n? Explain.

(502) Show that if n is an even perfect number, then $8n+1$ is a perfect square.

(503) Let a be a positive integer and p a prime number. Can p^a be a perfect number?

(504) Show that every even perfect number ends with the digit 6 or 8.

(505) Show that every even perfect number larger than 6 can be written as the sum of consecutive odd cubes.

(506) Show that each odd perfect number must have at least three distinct prime factors.

(507) Find all natural numbers n having the property that n and $\sigma(\sigma(n))$ are perfect numbers, or otherwise show that no such number n exists.

(508) A natural number n is said to be *tri-perfect* if $\sigma(n) = 3n$. Show that each odd tri-perfect number must be a perfect square.

(509) Show that the only tri-perfect numbers of the form $2^\alpha m$ with $1 \le \alpha \le 10$, m odd and $\mu^2(m) = 1$ are the numbers 120, 672, 523 776 and 459 818 240.

(510) A positive integer n is called respectively *deficient* or *abundant* if the sum of its divisors is $<$ or $>$ $2n$. Show that if the greatest common divisor of two positive integers a and b is deficient, then there exist

(a) infinitely many deficient numbers n such that $n \equiv a \pmod{b}$;

(b) infinitely many abundant numbers n such that $n \equiv a \pmod{b}$.

(511) Let n be an even perfect number. Show that

$$\tau(n) = [\log_2 n] + 2,$$

where $\log_2 n$ stands for the logarithm of n in basis 2.

(512) In 1997, Gordon Spence discovered the 36-th Mersenne prime, namely $2^{2976221} - 1$. Establish first a general formula allowing one to quickly compute the number of digits of a given large integer, and then use this formula to determine the number of digits contained in the prime number discovered by Spence.

(513) Let k be an arbitrarily large natural number. Prove that there exists an integer n such that

$$\frac{\sigma(n)}{n} > k.$$

(514) Show that an even perfect number is a triangular number, that is a number of the form $n(n+1)/2$.

(515) Show that a perfect number having k distinct prime factors has at least one prime factor which does not exceed k.

(516) Let $q_1, \ldots, q_k$ be distinct prime numbers. Show that

$$\frac{(q_1+1)(q_2+1)\cdots(q_k+1)}{q_1q_2\cdots q_k} \le 2 < \frac{q_1q_2\cdots q_k}{(q_1-1)(q_2-1)\cdots(q_k-1)}$$

is a necessary condition for $n = \prod_{i=1}^k q_i^{\alpha_i}$ to be a perfect number.

(517) Show that if n is an even perfect number, then $\phi(n) = 2^{k-1}(2^{k-1} - 1)$ for a certain positive integer k.

(518) Is it true that $\phi(n)$ is a multiple of 10 for infinitely many positive integers n?

(519) Calculate the number of positive integers ≤ 600 which have a factor > 1 in common with 600 and then count the number of positive integers ≤ 1200 which are relatively prime with 600.

(520) Count the number of positive integers ≤ 4200 which are relatively prime with 600 by observing that $4200 = 7 \cdot 600$.

(521) If m and k are positive integers, show that the number of positive integers $\le mk$ which are relatively prime with m is equal to $k\phi(m)$.

(522) Let $m, n \in \mathbb{N}$. Show that

$$\phi(mn) = \phi(m)\phi(n)\frac{d}{\phi(d)}, \qquad \text{where } d = (m,n).$$

(523) Show that for $n > 2$, $\phi(n)$ is an even number.

(524) Show that the number of fractions a/b, $(a,b) = 1$, such that $0 < a/b \le 1$, b being a fixed positive integer, is $\phi(b)$.

(525) Show that if $m|n$, then $\phi(m)|\phi(n)$.
(526) Identify all positive integers n such that $\phi(n)|n$.
(527) If $d|n$ and $k \in \mathbb{N}$, show that $\phi(nd^k) = d^k\phi(n)$.
(528) Identify all positive integers n such that $5\phi(n)|2n$.
(529) Characterize the set of positive integers n such that
(a) $\phi(2n) > \phi(n)$; (b) $\phi(2n) = \phi(n)$; (c) $\phi(2n) = \phi(3n)$.
(530) Let p be an odd prime number such that $2p+1$ is also a prime number. Show that if $n = 4p$, then $\phi(n+2) = \phi(n) + 2$.
(531) Show that for each integer $n \geq 2$, the sum of positive integers $\leq n$ and relatively prime with n is equal to $n\phi(n)/2$.
(532) Let $n > 1$ be an integer. Show that $2^{\omega(n)-1}|\phi(n)$.
(533) Show that $\phi(n)$ is a power of 2 if and only if $n = 2^\alpha F_1 \cdots F_r$, where $\alpha \geq 0$ and $F_i = 2^{2^i} + 1$, $i = 1, 2, \ldots, r$, are Fermat primes.
(534) Find the largest prime number p such that
(a) $p|\phi(95!)$; (b) $p^2|\phi(95!)$; (c) $p^3|\phi(95!)$; (d) $p^4|\phi(95!)$.
(535) Find the largest positive integer n such that $\phi(n) \leq 500$.
(536) Is it true that $\phi(8m+4) = 2\phi(4m+2)$ for each integer $m \geq 0$?
(537) Let $a, b \in \mathbb{N}$ such that $a|b$. Show that for each integer $n \geq 0$,

$$\frac{\phi(a^2n + ab)}{\phi(abn + a^2)} = \frac{\phi(an + b)}{\phi(bn + a)}.$$

(538) Show that $\phi(n) > n/7$ for all natural numbers n such that $\omega(n) \leq 9$.
(539) Given an odd integer $n > 3$, show that there exists a prime number p which divides $(2^{\phi(n)} - 1)$ but not n.
(540) Show that an integer $n \geq 2$ is prime if and only if $\phi(n)|(n-1)$ and $(n+1)|\sigma(n)$.
(541) Show that if e runs through the even divisors of n and d runs through the odd divisors of n, then

$$\frac{\displaystyle\sum_{e|n} \phi(n/e)}{\displaystyle\sum_{d|n} \phi(n/d)} = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

(542) Show that for $m > 2$,

$$\sum_{\substack{k=1 \\ (k,m)=1}}^{\phi(m)} \frac{1}{k}$$

is never an integer.
(543) Let $f(n)$ be the product of all the positive divisors of n. Does $f(m) = f(n)$ automatically implies that $m = n$?
(544) Let (i, n) be the greatest common divisor of the positive integers i and n. Express

$$\sum_{i=1}^{n} 2^{\omega((i,n))}$$

in terms of the prime factors appearing in the factorization of n.

(545) Let f be the arithmetical function defined by

$$f(n) = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 2 & \text{if } n \text{ is odd.} \end{cases}$$

Let $S(x) = \sum_{n \le x} f(n)$. Find the value of $\lim_{x\to\infty} \dfrac{S(x)}{x}$.

(546) Show that the expression $\sum_{p|n} \dfrac{1}{p}$ can become arbitrarily large if n is chosen appropriately.

(547) Show that

$$(*) \qquad \sum_{ab=n} f(a)f(b) = \tau(n)f(n) \qquad (n = 1, 2, \ldots)$$

if and only if $f(n)$ is totally multiplicative.

(548) Prove that for each positive integer n, we have

$$2^{\omega(n)} \le \tau(n) \le 2^{\Omega(n)}.$$

(549) Let

$$H(n) = \frac{\tau(n)}{\sum_{d|n} 1/d}$$

be the *harmonic mean* of the divisors of n. Show that n is an even perfect number if and only if $n = 2^{H(n)-1}(2^{H(n)} - 1)$.

(550) Show that

$$\tau(n)^2 = \sum_{c|n} \sum_{b|c} \sum_{a|b} \mu^2(a) \qquad (n = 1, 2, \ldots).$$

(551) Let $f\colon \mathbb{N} \to \mathbb{Z}$ be a function satisfying $f(n+m) \equiv f(n) \pmod{m}$ for all integers $m, n \ge 1$; any polynomial with integer coefficients is such a function. Let $g(n)$ be the number of values (counting repetitions) amongst $f(1), f(2), \ldots, f(n)$ which are divisible by n and let $h(n)$ be the number of these values which are relatively prime with n. Show that g and h are multiplicative functions and that

$$h(n) = \sum_{d|n} \mu(d)g(d)\,\frac{n}{d} = n \prod_{p|n} \left(1 - \frac{g(p)}{p}\right) \qquad (n = 1, 2, \ldots).$$

(552) Show that

$$\sum_{d|n} \lambda(d) = \begin{cases} 1 & \text{if } n = m^2 \text{ for a certain integer } m \ge 1, \\ 0 & \text{otherwise,} \end{cases}$$

where λ stands for the Liouville function.

(553) Let f be a multiplicative function such that $f(2) = 1$. Prove that if n is an even integer, then

$$\sum_{d|n} \mu(d)f(d) = 0.$$

(554) Let $f\colon [0,1]\cap\mathbb{Q}\to\mathbb{R}$ and let, for each integer $n\geq 1$,

$$F(n)=\sum_{k=1}^{n} f\left(\frac{k}{n}\right) \quad\text{and}\quad F^*(n)=\sum_{d|n}\sum_{\substack{1\leq k\leq d\\(k,d)=1}} f\left(\frac{k}{d}\right).$$

Show that

$$F(n)=\sum_{d|n} F^*(d) \qquad (n=1,2,\ldots).$$

(555) Given an integer m, let

$$\phi_m(n)=\sum_{\substack{k=1\\(k,n)=1}}^{n} k^m \qquad (n=1,2,\ldots).$$

Show, using the result of Problem 554, that

$$\sum_{d|n}\frac{\phi_m(d)}{d^m}=\frac{1^m+2^m+\cdots+n^m}{n^m} \qquad (n=1,2\ldots).$$

Use this equation to show that

$$\sum_{\substack{k=1\\(k,n)=1}}^{n} k^m=\sum_{d|n} d^m\mu(d)\left(1^m+2^m+\cdots+\left(\frac{n}{d}\right)^m\right) \qquad (n=1,2\ldots).$$

(556) Prove that for each positive integer n,

$$\sum_{\substack{k=1\\(k,n)=1}}^{n} k=\frac{n}{2}\phi(n)+\frac{n}{2}\sum_{d|n}\mu(d).$$

(557) Prove that for each positive integer n,

$$\sum_{\substack{k=1\\(k,n)=1}}^{n} k^2=\frac{n^2}{3}\phi(n)+\frac{n^2}{2}\sum_{d|n}\mu(d)+\frac{n}{6}\prod_{p|n}(1-p).$$

(558) Prove that for each positive integer n,

$$\sum_{\substack{k=1\\(k,n)=1}}^{n} k^3=\frac{n^3}{4}\phi(n)+\frac{n^3}{2}\sum_{d|n}\mu(d)+\frac{n^2}{4}\prod_{p|n}(1-p).$$

(559) For each positive integer n, set $f(n)=\sum_{d|n}\frac{\mu^2(d)}{\tau(d)}$. Establish a formula for $f(n)$ in terms of the canonical representation of n.

(560) Is it true that $n=\sum_{d|n}\mu(d)\sigma(n/d)$ for each positive integer n?

(561) Show that, for each positive integer n,

$$\sum_{d|n}\frac{1}{d^2}=\frac{\sigma_2(n)}{n^2}.$$

(562) Show that $\sum_{d|n} \tau^3(d) = \left(\sum_{d|n} \tau(d)\right)^2$ for each positive integer n.

(563) Show that each of the following relations holds for each integer $n \geq 1$:

$$\sum_{d|n} |\mu(d)| = 2^{\omega(n)}; \quad \sum_{d|n} \mu(d)\tau(d) = (-1)^{\omega(n)}; \quad \sum_{d|n} \mu(d)\lambda(d) = 2^{\omega(n)};$$

$$\sum_{d|n} \mu(d)\sigma(d) = (-1)^{\omega(n)} \prod_{p|n} p \text{ and } \sum_{d|n} \mu(d)\phi(d) = (-1)^{\omega(n)} \prod_{p|n} (p-2).$$

(564) Show that, for each positive integer n,

$$\frac{n}{\phi(n)} = \sum_{d|n} \frac{\mu^2(d)}{\phi(d)}.$$

(565) Let g be a multiplicative function. For each positive integer n, set

$$F(n) = \sum_{d|n} \mu(d) g(n/d)$$

and show that

$$F(n) = \prod_{p^\alpha \| n} \left(g(p^\alpha) - g(p^{\alpha-1}) \right).$$

(566) Let f be the function defined by

$$f(n) = \sum_{[d,r]=n} \phi(d)\phi(r) \qquad (n = 1, 2, \ldots),$$

where the sum runs over all ordered pairs (d, r) such that $[d, r] = n$ (see Problem 468). Show that

$$f(n) = n^2 \prod_{p|n} \left(1 - \frac{1}{p^2}\right).$$

(567) Let Λ be the von Mangoldt function. Show that

$$\sum_{d|n} \Lambda(d) = \log n \qquad (n = 1, 2, \ldots).$$

(568) Prove that for each positive integer n,

$$\Lambda(n) = \sum_{d|n} \mu(d) \log(n/d).$$

(569) Let f be a multiplicative function. Show that

$$\sum_{d|n} (-1)^{n/d} f(d) =$$

$$\begin{cases} -\sum_{d|n} f(d), & \text{if } n \text{ is odd,} \\ \sum_{d|n} f(d) - 2f(2^k) \sum_{d|m} f(d), & \text{if } n = 2^k m,\ (m,2) = 1,\ k \geq 1. \end{cases}$$

(570) Show that if n is an even integer, then

$$\sum_{d|n}(-1)^{n/d}\phi(d) = 0.$$

What if n is an odd integer?

(571) Let f be an arithmetical function verifying the equation $\sum_{d|n} f(d) = n$ for each positive integer n. Show that $f(n) = \phi(n)$ for each $n \geq 1$.

(572) For each function g defined implicitly above, find a formula for $g(n)$ in terms of the canonical representation of n:

(a) $n^2 = \sum_{d|n} g(d)$; (b) $\mu(n) = \sum_{d|n} g(d)$.

(573) Show that the function $2^{\omega(n)}n/\phi(n)$ is multiplicative and find a formula for $g(n)$ in terms of the canonical representation of n, knowing that

$$2^{\omega(n)}\frac{n}{\phi(n)} = \sum_{d|n} g(d), \qquad \text{for each integer } n \geq 1.$$

(574) For each positive integer n, show that

$$\sum_{d|n}(-1)^{\Omega(d)}\mu(d) = 2^{\omega(n)}, \quad \sum_{d|n}(-1)^{\Omega(d)}\mu\left(\frac{n}{d}\right) = (-1)^{\Omega(n)}2^{\omega(n)}$$

and that

$$\sum_{d|n}(-1)^{\Omega(d)}2^{\omega(n/d)} = 1.$$

(575) Show that, for each positive integer n,

$$\sum_{d|n}\mu(d)\lambda\left(\frac{n}{d}\right) = (-1)^{\Omega(n)}\,2^{\omega(n)}.$$

(576) Let g be an arithmetical function such that $g(n) > 0$ for each positive integer n and let

$$f(n) = \prod_{d|n} g(d) \qquad (n = 1, 2, \ldots).$$

Show that

$$g(n) = \prod_{d|n}(f(n/d))^{\mu(d)} \qquad (n = 1, 2, \ldots).$$

(577) Let k be a real number. Show that, for each positive integer n,

$$\prod_{d|n} d^{(k/2)\tau(d)\mu(n/d)} = n^k.$$

(578) Let f be a totally multiplicative function and let F be defined by

$$F(n) = \sum_{d|n} f(d) \qquad (n = 1, 2, \ldots).$$

Do we necessarily have that F is also totally multiplicative?

(579) Prove that for each integer $n \geq 1$,

$$\sum_{\substack{d|n \\ \mu^2(d)=1}} \mu(n/d) = \begin{cases} \mu(\sqrt{n}) & \text{if } n = m^2, \\ 0 & \text{otherwise.} \end{cases}$$

(580) Let f be an arithmetical function. Show that for each integer $n \geq 1$,

$$\prod_{d|n} d^{f(d)+f(n/d)} = n^{\sum_{d|n} f(d)}.$$

Use this to show that

$$(*) \qquad \prod_{d|n} d^{\phi(d)+\phi(n/d)} = n^n \qquad (n = 1, 2, \ldots).$$

(581) Letting as usual $f * g$ stand for the Dirichlet product of the arithmetical functions f and g, show that if A stands for the set of arithmetical functions $f\colon \mathbb{N} \to \mathbb{R}$ such that $f(1) \neq 0$, then A is a commutative group with respect to the operation $*$; thus, prove successively that:

(a) the Dirichlet product is commutative; that is if f and g are arithmetical functions, then $f * g = g * f$;

(b) the Dirichlet product is associative; that is if f, g and h are arithmetical functions, then $(f * g) * h = f * (g * h)$;

(c) the arithmetical function E defined by $E(1) = 1$ and $E(n) = 0$ for $n > 1$ is such that $f * E = E * f = f$ for each arithmetical function f;

(d) for each arithmetical function f such that $f(1) \neq 0$, there exists a function f^{-1} called the *inverse function* of f (with respect to the Dirichlet product $*$) such that $f^{-1} * f = f * f^{-1} = E$ and that f^{-1} is given by the recurrence formula

$$f^{-1}(1) = \frac{1}{f(1)}, \qquad f^{-1}(n) = -\frac{1}{f(1)} \sum_{\substack{d|n \\ d<n}} f\left(\frac{n}{d}\right) f^{-1}(d) \qquad \text{for } n > 1.$$

(582) Let f and g be two arithmetical functions. Show that if g and $f * g$ are multiplicative, then f is also multiplicative.

(583) Show that if f is a multiplicative function such that $f(1) \neq 0$, then its inverse f^{-1}, with respect to the Dirichlet product $*$, is also multiplicative.

(584) Let r be a real number and let ι_r be the arithmetical function defined by $\iota_r(n) = n^r$ for each positive integer n. Show that

(a) $\mu * \iota_0 = E$, where μ is the Moebius function.

(b) $\sigma_r = \iota_r * \iota_0$, where $\sigma_r(n) = \sum_{d|n} d^r$.

(c) $\phi = \iota_1 * \mu$, where ϕ is Euler's function.

(d) $\mu * \sigma = \iota_1$, where $\sigma = \sigma_1$.

(e) $\phi * \sigma_r = \iota_1 * \iota_r$.

(f) $\iota_1 * \iota_1 = \iota_1 \tau$, where $\tau = \sigma_0$.

(g) $f = \iota_0 * \iota_0 * \iota_0$, where $f(n) = \sum_{d|n} \tau(d)$.

(585) Show that $\iota_0^{-1} = \mu$ and more generally that $\iota_r^{-1} = \mu\, \iota_r$ for each real number r.

(586) For each of the arithmetical functions f given below, determine its inverse f^{-1} (with respect to the Dirichlet product $*$):

(i) $f(n) = \iota_0(n)$; (ii) $f(n) = E(n)$; (iii) $f(n) = |\mu(n)|$.

(587) Let f and g be two arithmetical functions such that $f(1) \neq 0$ and $g(1) \neq 0$. Show that

$$(f * g)^{-1} = f^{-1} * g^{-1}.$$

(588) Let ϕ^{-1} be the inverse (with respect to the Dirichlet product $*$) of the Euler function ϕ. Show that

$$\phi^{-1}(n) = \prod_{p|n}(1-p) \qquad (n = 1, 2, \ldots).$$

(589) Let f be a multiplicative function. Show that f is totally multiplicative if and only if $f^{-1}(n) = \mu(n)f(n)$ for each integer $n \geq 1$.

(590) Show that the inverse, with respect to the Dirichlet product $*$, of the Liouville function λ is

$$\lambda^{-1}(n) = \lambda(n)\mu(n) = \begin{cases} 1 & \text{if } n \text{ is squarefree,} \\ 0 & \text{otherwise.} \end{cases}$$

(591) Show that the inverse, with respect to the Dirichlet product $*$, of σ_a is given by

$$\sigma_a^{-1}(n) = \sum_{d|n} d^a \mu(d)\mu(n/d) \qquad (n = 1, 2, \ldots).$$

(592) Let m and n be positive integers. Show that

$$\sigma(n)\sigma(m) = \sum_{d|(m,n)} d\sigma(mn/d^2).$$

(593) Let f be a multiplicative function. Show that the following three statements are equivalent:

(a) There exists a multiplicative function F such that for all positive m and n,

(1) $$f(mn) = \sum_{d|(m,n)} f(m/d)f(n/d)F(d).$$

(b) There exists a totally multiplicative function g such that for all integers m and n,

(2) $$f(m)f(n) = \sum_{d|(m,n)} f(mn/d^2)\, g(d).$$

(c) For each prime number p and each integer $a \geq 1$,

(3) $$f(p^{a+1}) = f(p)f(p^a) + f(p^{a-1})\Big(f(p^2) - f^2(p)\Big).$$

(594) Show that, for each positive integer n,

$$\sum_{d|n} \sigma(d) = n \sum_{d|n} \frac{\tau(d)}{d}.$$

(595) Show that, for each positive integer n,

$$\sum_{d|n} \phi(d)\tau\left(\frac{n}{d}\right) = \sigma(n).$$

(596) Show that, for each positive integer n,

$$\sum_{d|n} \phi(d)\sigma\left(\frac{n}{d}\right) = n\tau(n).$$

(597) Show that, for each positive integer n,

$$\sum_{d|n} \frac{n}{d}\sigma(d) = \sum_{d|n} d\tau(d).$$

(598) Show that, for each positive integer n,

$$\sum_{d|n} d\sigma(d) = \sum_{d|n} \left(\frac{n}{d}\right)^2 \sigma(d).$$

(599) Let k and r be real numbers. Show that

$$\sum_{d|n} d^r \sigma_{k-r}(d) = \sum_{d|n} \left(\frac{n}{d}\right)^k \sigma_r(d).$$

(600) Show that, for each positive integer n,

$$\sum_{d|n} \sigma(d)\sigma\left(\frac{n}{d}\right) = \sum_{d|n} d\,\tau(d)\tau\left(\frac{n}{d}\right).$$

(601) Let r be a real number. Show that

$$\sum_{d|n} \sigma_r(d)\sigma_r\left(\frac{n}{d}\right) = \sum_{d|n} d^r\tau(d)\tau\left(\frac{n}{d}\right).$$

(602) Show that, for each positive integer n,

$$\sum_{d|n} \mu(d)\tau(n/d) = 1.$$

(603) Show that $\Lambda = \mu * \log$, where Λ is the von Mangoldt function.

(604) Show that $\sum_{d|n} \mu^2(d)\Lambda(d) = \log\gamma(n)$ for each positive integer n, where $\gamma(n) = \prod_{p|n} p$ and $\gamma(1) = 1$ and Λ stands for the von Mangoldt function.

(605) Let f be a totally arithmetical function which only takes the values $+1$ and -1. Let $I = [N, N+M]$, where $M \geq 3\sqrt{N}$. Assume that there exists an integer $n_0 < \sqrt{N}$ such that $f(n_0) = -1$. Prove that this function f cannot be constant on the interval I.

(606) Show that the Liouville function λ does necessarily take the two values $+1$ and -1 on any interval of the form $[N, N+3\sqrt{N}]$, $N \geq 2$.

(607) Given an arbitrary real number $x \geq 1$, show that

$$\sum_{1\leq d\leq x} \mu(d)\left[\frac{x}{d}\right] = 1.$$

(608) Given an arbitrary real number $x \geq 1$, show that

$$\left| \sum_{1 \leq n \leq x} \frac{\mu(n)}{n} \right| \leq 1.$$

(609) Let $\delta(n)$ be the largest odd divisor of the positive integer n. Show that for each integer $m \geq 1$,

$$\left| \sum_{n=1}^{m} \frac{\delta(n)}{n} - \frac{2m}{3} \right| < 1.$$

(610) It is clear that any positive integer n can be written uniquely in the form $n = mr^2$ where m is squarefree. In light of this, justify the chain of equations

$$\mu^2(n) = \mu^2(mr^2) = E(r) = \sum_{d|r} \mu(d) = \sum_{d^2|n} \mu(d).$$

(611) Let f be a strongly multiplicative function such that $0 \leq f(p) \leq 1$ for each prime number p and such that $f(2) = f(3) = 0$. Show that, for each positive integer N,

$$\sum_{n \leq N} f(n) \leq \frac{N}{3} + 2.$$

(612) Is it possible to construct a multiplicative function f such that $f(2) = 0$ and such that, as $N \to \infty$,

$$\sum_{n \leq N} f(n) \sim \frac{3N}{4} \ ?$$

(613) Establish that the number $A(N)$ of squarefree integers $\leq N$ satisfies the relation

$$A(N) = \sum_{d \leq \sqrt{N}} \mu(d) \left[\frac{N}{d^2} \right].$$

(614) Let $\phi(n)$ be Euler's function and let $\tau(n)$ be the function which counts the number of positive divisors of n. Show that

$$\liminf_{n \to \infty} \frac{\phi(n)}{n\tau(n)} = 0 \quad \text{and} \quad \limsup_{n \to \infty} \frac{\phi(n)}{n\tau(n)} = \frac{1}{2}.$$

(615) Consider the sequence $u_n = 2^n$, $n \geq 1$. For each positive integer n, choose the smallest prime number q_n satisfying $u_n \leq q_n < u_{n+1}$; according to Bertrand's Postulate, such a prime number exists. What can be said about the convergence or the divergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{q_n}$? Explain.

(616) Prove that for each positive integer n, $\tau(2^n + 1) > \tau_1(n)$, where $\tau_1(n)$ stands for the number of odd divisors of n.

(617) Let $n > 1$ be a composite number. Show that $\sigma(n) > n + \sqrt{n}$. Use this to prove that $\displaystyle\lim_{n \to \infty} (\sigma(p_n + 1) - \sigma(p_n)) = +\infty$.

(618) Show that, for each positive integer n, we have

$$\frac{\sigma(n)}{n} < \begin{cases} \left(\frac{3}{2}\right)^{\omega(n)} & \text{if } n \text{ is odd,} \\ 2\left(\frac{3}{2}\right)^{\omega(n)-1} & \text{if } n \text{ is even.} \end{cases}$$

(619) Show that $\sigma(n) < n\tau(n)$ for each integer $n \geq 2$.
(620) Find infinitely many integers n such that $\sigma(n) \leq \sigma(n-1)$.
(621) Show that for each integer $n \geq 1$, $n \leq \sigma(n) \leq n^2$.
(622) Let $\sigma_p(n)$ stand for the sum of even divisors of the integer $n \geq 1$. Show that

$$\sigma_p(n) \geq \alpha\tau(m)\sqrt{2n},$$

where α and m are defined implicitly by $n = 2^\alpha m$, m odd.
(623) Prove that for each integer $n \geq 2$, $\sigma(n) \geq \phi(n) + \tau(n)$, with equality if and only if n is prime.
(624) Let f and g be two multiplicative functions taking only positive values. Show that for each integer $n \geq 2$,

$$\sum_{d|n} f(d)g(n/d) \geq f(n) + g(n),$$

with equality if and only if n is prime.
(625) Show that for each integer $n \geq 2$, we have $\sigma(n) + \phi(n) \leq n\tau(n)$, with equality if and only if n is prime.
(626) Let f, g and h be three multiplicative functions. If for each integer $n \geq 1$, $f(n) + g(n) \geq 0$ and $h(n) \geq 0$, show that,

$$\sum_{d|n} f(d)h(n/d) + \sum_{d|n} g(d)h(n/d) \geq 2h(n) \quad (n = 1, 2, \ldots),$$

with equality if and only if $n = 1$ or else for each $d|n$, $d > 1$, $f(d)+g(d) = 0$.

(627) Show that for each integer $n \geq 1$, $\sigma(n) + \phi(n) \geq 2n$, with equality if and only if $n = 1$ or n is prime.
(628) Prove that for each integer $n \geq 2$, we have $\sigma(n) > n + (\omega(n) - 1)\sqrt{n}$.
(629) Show that for each integer $n > 2$, we have

$$\phi(n^2) + \phi((n+1)^2) < 2n^2.$$

More generally, show that for all integers $n > 2$ and $k \geq 2$,

$$\phi(n^k) + \phi((n+1)^k) < 2n^2(n+1)^{k-2}.$$

(630) Show that

$$\limsup_{n\to\infty} \frac{\phi(n+1)}{\phi(n)} = \infty \quad \text{and} \quad \liminf_{n\to\infty} \frac{\phi(n+1)}{\phi(n)} = 0.$$

(631) Can one find arbitrarily large integers N such that $\phi(n) \geq \phi(N)$ for each integer $n \geq N$, while $\phi(n) \leq \phi(N)$ for each integer $n \leq N$?
(632) Show that

$$\phi(n)\tau^2(n) \leq n^2$$

for all the positive integers $n \neq 4$. For which values of n does equality hold?
(633) Show that, for each integer $n \geq 2$, $\phi(n) \leq n - n^{1-\frac{1}{\omega(n)}}$, with equality if and only if n is prime.

(634) If n is a composite integer, show that $\phi(n) \leq n - \sqrt{n}$.

(635) Show that if $\omega(n) = r$ for positive integers n and r, then

$$\phi(n) \geq \frac{n}{2^r}.$$

(636) For each $n \in \mathbb{N}$, let $\sigma(n) = \sum_{d|n} d$ and $\sigma_2(n) = \sum_{d|n} d^2$. Show that

$$\frac{\sigma^2(n)}{\tau(n)} \leq \sigma_2(n) \leq \sigma^2(n) \qquad (n = 1, 2, \ldots).$$

(637) Show that the mean value of the divisors of the positive integer n is larger or equal to $\prod_{d|n} d^{1/\tau(n)}$.

(638) Show that

$$\frac{\sigma(n)}{\tau(n)} \geq \sqrt{n} \qquad (n = 1, 2, \ldots).$$

(639) Show that $\prod_{d|n} d = n^3$ if and only if $n = p^5$ or $n = p^2 q$, with p and q distinct prime numbers.

(640) For each integer $n \geq 1$, show that $\tau(n) \leq 2\sqrt{n}$.

(641) Prove that for each integer $n \geq 3$, we have $\sigma(n) < n\sqrt{n}$.

(642) Prove that for each integer $n \geq 2$,

$$\sum_{\substack{d|n \\ d \geq 2}} \frac{d-1}{\log d} > \frac{2\tau(n)(\sqrt{n}-1)}{\log n} - 1.$$

(643) Prove that for each integer $n \geq 2$,

$$\sum_{\substack{d|n \\ d \geq 2}} \frac{d^2-1}{\log d} > \frac{2\tau(n)(n-1)}{\log n} - 2.$$

(644) Prove that for each integer $n \geq 2$,

$$\prod_{p|n} \left(1 - \frac{1}{p}\right) \leq 1 - \frac{2^{\omega(n)} - 1}{n},$$

with equality if and only if n is prime.

(645) Prove that for each integer $n \geq 2$,

$$\sum_{p|n} \frac{1}{p} \geq \frac{\omega(n)}{n^{1/\omega(n)}}.$$

(646) Let h be the arithmetical function defined by $h(1) = 0$ and $h(p^a) = 0$ if p is prime and a a positive integer, and otherwise, that is if $n = q_1^{a_1} \cdots q_r^{a_r}$, with $r \geq 2$, q_i prime, by

$$h(n) = \sum_{i=2}^{\omega(n)} \frac{1}{q_i - q_{i-1}}.$$

For each positive integer n such that $\omega(n) \geq 2$, show that

$$h(n) \geq \frac{(\omega(n)-1)^2}{P(n) - p(n)},$$

where $p(n)$ and $P(n)$ stand respectively for the smallest and largest prime factors of n.

(647) Let H be the arithmetical function defined by $H(1) = 0$, and for $n > 1$ by

$$H(n) = \sum_{i=2}^{\tau(n)} \frac{1}{d_i - d_{i-1}},$$

where $1 = d_1 < d_2 < \ldots < d_{\tau(n)} = n$ represent the divisors of n. Show that, for each positive integer n,

$$H(n) \geq \frac{(\tau(n) - 1)^2}{n - 1}.$$

(648) Show that $\tau(2^n - 1) \geq \tau(n)$ for each integer $n \geq 1$.

(649) Let m and n be positive integers; show that $\phi(mn) \leq m\phi(n)$. On the other hand, if each prime number dividing m also divides n, then show that $\phi(mn) = m\phi(n)$.

(650) For each integer $n \geq 1$, show that

$$\phi\left(n\left[\frac{\sigma(n)\,\tau(n)}{n^{3/2}}\right]\right) \leq 2n,$$

where $[y]$ stands for the largest integer $\leq y$.

(651) Show that $\phi(n)\tau(n) \geq n$ for each positive integer n.

(652) Find all the solutions of the equation $\phi(n)\tau(n) = n$, where $n \in \mathbb{N}$.

(653) Let m and n be integers larger than 2; show that $\phi(mn) + \phi((m+1)(n+1)) < 2mn$.

(654) Consider the arithmetical function $\Psi(n)$ defined by

$$\Psi(n) = n\prod_{p|n}\left(1 + \frac{1}{p}\right) \qquad (n = 1, 2 \ldots).$$

It is clear that the function Ψ is multiplicative.

(a) Show that $\Psi(n) \leq \sigma(n)$, where $\sigma(n)$ represents the sum of the divisors of n.

(b) Show that $\Psi(n) = \sigma(n)$ if and only if n is squarefree.

(c) We say that a natural number n is Ψ-perfect if $\Psi(n) = 2n$. Prove that a number n is Ψ-perfect if and only if it is of the form $2^a \cdot 3^b$, where a and b are positive integers.

(655) Let f be a polynomial with integer coefficients and let

$$\phi^*(n) = \#\{k \mid 1 \leq k \leq n, (f(k), n) = 1\}.$$

Observe that in the case $f(n) = n$, we find that $\phi^*(n) = \phi(n)$, that is Euler's function.

(a) Show that ϕ^* is a multiplicative function.

(b) Show that, for each positive integer n,

$$\phi^*(n) = n\prod_{p|n}\left(1 - \frac{b_p}{p}\right),$$

where $b_p = p - \phi^*(p) = \#\{k \mid 1 \leq k \leq p,\ p|f(k)\}$.

(656) Let n be a positive integer. Find the number of terms of the sequence

$$1 \cdot 2,\ 2 \cdot 3,\ 3 \cdot 4, \ldots, n(n+1)$$

which are relatively prime with n.

(657) Let n be a positive integer. Find a formula which gives the number of positive integers $k \leq n$ such that $(k,n) = (k+1,n) = 1$.

(658) Let n be a positive integer. Find an expression for the number of terms of the sequence

$$1 \cdot 2 \cdot 3,\ 2 \cdot 3 \cdot 4, \ldots, n(n+1)(n+2)$$

which are relatively prime with n.

(659) Let $f(n)$ be the n-th positive integer which is not a perfect square. Hence $f(1) = 2$, $f(2) = 3$, $f(3) = 5$ and $f(4) = 6$. Show that, for each integer $n \geq 1$, $f(n) = n + \|\sqrt{n}\|$, where $\|x\|$ stands for the closest integer to x.

(660) Show that

$$\sum_{d^k|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 1 & \text{if } p^\alpha | n \Rightarrow \alpha < k, \\ 0 & \text{otherwise,} \end{cases}$$

thus generalizing the result of Problem 610.

(661) Prove that the Liouville function λ takes infinitely many times each of the values $+1, -1$ when applied to the sequence of integers $2, 5, 10, 17, \ldots, n^2 + 1, \ldots$.

(662) Let n be a positive integer. If $\{a_1, a_2, \ldots, a_k\}$ is the set of positive integers $i \leq n$ with $(i,n) = 1$, show that

$$\sum_{i=1}^{k} \frac{a_i}{n - a_i} \geq \phi(n) \qquad (n = 1, 2, \ldots).$$

(663) Let $f\colon \mathbb{N} \to \mathbb{R}^+$ and let $F(n) = \sum_{d|n} f(d)$. Show that

$$\prod_{d|n} f(d) \leq \left(\frac{F(n)}{\tau(n)} \right)^{\tau(n)} \qquad (n = 1, 2, \ldots).$$

Use this result to show that

$$\prod_{d|n} \phi(d) \leq \left(\frac{n}{\tau(n)} \right)^{\tau(n)} \qquad (n = 1, 2, \ldots).$$

(664) Show that there exists a positive constant C such that, for each integer $n \geq 2$,

$$C < \frac{\sigma(n)\phi(n)}{n^2} < 1.$$

(665) Define the *derivative* f' of an arithmetical function f by

$$f'(n) = f(n) \log n \qquad (n = 1, 2 \ldots).$$

Show that, given any arithmetical functions f and g, we have

(a) $(f+g)' = f' + g'$,
(b) $(f * g)' = f' * g + f * g'$,
(c) $(f^{-1})' = -f' * (f * f)^{-1}$ provided that $f(1) \neq 0$.

(666) Given an arithmetical function f, define

$$(*) \qquad \overline{f}(n) = \frac{1}{\tau(n)} \sum_{d|n} f(d) \qquad (n = 1, 2, \ldots),$$

where $\tau(n)$ stands for the number of divisors of n. Show that if f is multiplicative, then $\overline{f}$ is also multiplicative.

(667) Let $\overline{f}$ be the function introduced in Problem 666. Show that if f is additive, then $\overline{f}$ is also additive.

(668) Let $1(n) = 1$ for each positive integer n and let μ be the Moebius function. What represents the functions $\overline{1}$ and $\overline{\mu}$, where $\overline{f}$ is defined by the relation $(*)$ of Problem 666?

(669) Let $\omega(n) = \sum_{p|n} 1$. Determine the values of the function $\overline{\omega}$, where $\overline{f}$ is defined by the relation $(*)$ of Problem 666.

(670) Let $f(n) = 2^{\omega(n)}$, where $\omega(n) = \sum_{p|n} 1$. Prove that

$$\overline{f}(n) = \frac{\tau(n^2)}{\tau(n)} \qquad (n = 1, 2, \ldots),$$

where $\overline{f}$ is defined by the relation $(*)$ of Problem 666.

(671) Let $f(n) = 2^{\Omega(n)}$, where $\Omega(n) = \sum_{p^\alpha \| n} \alpha$. Show that

$$\overline{f}(n) = \prod_{p^\alpha \| n} \frac{2^{\alpha+1} - 1}{\alpha + 1} \qquad (n = 1, 2, \ldots),$$

where $\overline{f}$ is defined by the relation $(*)$ of Problem 666.

(672) Let λ be the Liouville function. Show that $\overline{\lambda}(n)\tau(n) = \chi(n)$, where $\chi(n)$ is the characteristic function of the set of perfect squares, that is

$$\chi(n) = \begin{cases} 1 & \text{if } n = m^2, \\ 0 & \text{otherwise,} \end{cases}$$

and where $\overline{f}$ is defined by the relation $(*)$ of Problem 666.

(673) Given a multiplicative function g, show that there exists a multiplicative function f such that $g = \overline{f}$, where $\overline{f}$ is defined by the relation $(*)$ of Problem 666.

(674) Let $g(n) = 2^{\omega(n)}$. Find the function f such that $g = \overline{f}$, where $\overline{f}$ is defined by the relation $(*)$ of Problem 666.

(675) Given an arithmetical function f, define

$$(**) \qquad \widehat{f}(n) = \frac{1}{2^{\omega(n)}} \sum_{d|n} \mu^2(d) f(d) \qquad (n = 1, 2, \ldots),$$

where $\omega(n) = \sum_{p|n} 1$ and μ stands for the Moebius function. Show that if f is multiplicative, then $\widehat{f}$ is multiplicative.

(676) Let $1(n) = 1$ for each positive integer n and λ stand for the Liouville function. Determine the functions $\widehat{1}$ and $\widehat{\lambda}$, where $\widehat{f}$ is defined by the relation $(**)$ of Problem 675.

(677) We know that $\tau(n)$ represents the number of ways of writing a positive integer n as a product of two positive integers, taking into account the

order of the factors. In other words,

$$\tau(n) = \sum_{d_1 d_2 = n} 1.$$

More generally, given an integer $k \geq 2$, let $\tau_k(n)$ be the number of ways of writing a positive integer n as a product of k positive integers, taking into account the order of the factors. In other terms,

$$\tau_k(n) = \sum_{d_1 d_2 \cdots d_k = n} 1.$$

Show that

$$\tau_k = \underbrace{1 * 1 * \ldots * 1}_{k}.$$

(678) Show that if $F(k) = \sum_{d|k} f(d)$ for $k = 1, 2, \ldots$, then $\sum_{k=1}^{n} F(k) = \sum_{k=1}^{n} \left[\frac{n}{k}\right] f(k)$ for each positive integer n.

(679) Show that

$$\sum_{k=1}^{2n} \tau(k) - \sum_{k=1}^{n} \left[\frac{2n}{k}\right] = n \qquad (n = 1, 2, \ldots).$$

(680) Show that $\sum_{d=1}^{n} \phi(d) \left[\frac{n}{d}\right] = \frac{1}{2} n(n+1)$ for each positive integer n.

(681) Show that $\sum_{k=1}^{n} \Lambda(k) \left[\frac{n}{k}\right] = \log n!$ for each positive integer n.

(682) Show that $\sum_{k=1}^{n} \lambda(k) \left[\frac{n}{k}\right] = [\sqrt{n}]$ for each positive integer n.

(683) Given an integer $n \geq 2$ and p a prime divisor of n, let p^{a_p} be the largest power of p not exceeding n, meaning that a_p is the only positive integer satisfying $p^{a_p} \leq n < p^{a_p+1}$. Finally, let

$$S(n) = \sum_{p|n} p^{a_p} \qquad (n = 2, 3, \ldots).$$

Show that there exist infinitely many integers n such that $S(n) > n$.

(684) Let n be a positive integer. Show that

$$\sum_{k=1}^{n} \tau(k) = \sum_{k=1}^{n} \left[\frac{n}{k}\right].$$

(685) Let n be a positive integer. Show that $\sum_{k=1}^{n} \sigma(k) = \sum_{k=1}^{n} k \left[\frac{n}{k}\right]$.

(686) Let n be a positive integer. Show that

$$\sum_{k=1}^{n} \phi(k) = \frac{1}{2} \sum_{k=1}^{n} \mu(k) \left[\frac{n}{k}\right] \left(\left[\frac{n}{k}\right] + 1\right).$$

(687) Let n be a positive integer. Show that

$$\sum_{k=1}^{n} k \sum_{d|k} \lambda(d) = \frac{[\sqrt{n}]\Big([\sqrt{n}]+1\Big)\Big(2[\sqrt{n}]+1\Big)}{6}.$$

(688) For each positive integer n, let $S(n)$ be the set of all positive integers k such that the fractional part of n/k is $\geq 1/2$. Let f be an arbitrary arithmetical function and let

$$g(n) = \sum_{k=1}^{n} f(k)\left[\frac{n}{k}\right] \qquad (n = 1, 2, \ldots).$$

Show that

$$\sum_{k \in S(n)} f(k) = g(2n) - 2g(n) \qquad (n = 1, 2, \ldots).$$

Use this result to show that for each integer $n \geq 1$,

$$\sum_{k \in S(n)} \phi(k) = n^2, \qquad \sum_{k \in S(n)} \mu(k) = -1,$$

$$\sum_{k \in S(n)} \Lambda(k) = \log\left(\binom{2n}{n}\right), \qquad \sum_{k \in S(n)} \lambda(k) = [\sqrt{2n}] - 2[\sqrt{n}],$$

where ϕ, μ, Λ and λ are respectively the Euler function, the Moebius function, the von Mangoldt function and the Liouville function.

(689) Let x be a real number such that $|x| < 1$. Show that

$$\sum_{n=1}^{\infty} \frac{\phi(n)\, x^n}{1 - x^n} = \frac{x}{(1-x)^2} \qquad (n = 1, 2, \ldots),$$

where ϕ is the Euler function.

(690) Let f and g be two arithmetical functions tied by the relation $f(n) = \sum_{d|n} g(d)$, $n \geq 1$, and x a real number such that $|x| < 1$. Show that

$$\sum_{n=1}^{\infty} g(n) \frac{x^n}{1 - x^n} = \sum_{n=1}^{\infty} f(n)\, x^n \qquad (n = 1, 2, \ldots).$$

(691) Let n be a positive integer. Consider the square matrix $M_{n \times n} = (b_{ij})_{n \times n}$, where the element $b_{ij} = (i, j)$, that is the GCD of i and j. Use the fact that $\sum_{d|n} \mu(d) \frac{n}{d} = \phi(n)$ in order to prove that

$$\det M = \phi(1)\phi(2)\cdots\phi(n),$$

where ϕ stands for Euler's function.

(692) Let n be a positive integer and let $M = (a_{ij})_{n \times n}$ be the matrix whose a_{ij} element is defined by $a_{ij} = \tau((i, j))$, where $\tau(m)$ represents the number of divisors of m. Show that $\det M = 1$.

(693) Let n be a positive integer and let $M = (a_{ij})_{n \times n}$ be the matrix whose a_{ij} element is defined by $a_{ij} = \sigma((i, j))$, where $\sigma(m)$ represents the sum of the divisors of m. Show that $\det M = n!$.

(694) Let n be a positive integer and let $M = (a_{ij})_{n\times n}$ be the matrix whose a_{ij} element is defined by $a_{ij} = \mu((i,j))$, where μ stands for the Moebius function. Show that $\det M \neq 0$ for $1 \le n \le 7$ and then that $\det M = 0$ for $n \ge 8$.

(695) Let n be a positive integer and let $M = (a_{ij})_{n\times n}$ be the matrix whose a_{ij} element is defined by $a_{ij} = [i,j]$, that is the LCM of i and j. Show that

$$\det M = \prod_{k=1}^{n} (-1)^{\omega(k)} \phi(k)\gamma(k),$$

where $\omega(k) = \sum_{p|k} 1$, $\gamma(k) = \prod_{p|k} p$ and ϕ stands for Euler's function.

(696) Let k be a positive integer and let f be an arithmetical function. Show that if

$$g(x) = \sum_{\substack{n\le x\\(n,k)=1}} f(x/n),$$

then

$$f(x) = \sum_{\substack{n\le x\\(n,k)=1}} \mu(n) g(x/n).$$

(697) Let f be an arithmetical function. Show that

$$\sum_{\substack{n\le N\\(n,k)=1}} f(n) = \sum_{d|k} \sum_{m\le N/d} \mu(d) f(md).$$

(698) Let $M(x) := \sum_{n\le x} \mu(n)$. Show that

$$\sum_{n\le x} M(x/n) = 1.$$

(699) Let $p(n)$ be the smallest prime factor of n, $p(1) = 1$. Show that

$$\sum_{n\le x} p(n(n+1)) = 2[x].$$

(700) Recently, when Canada celebrated its 125^{th} anniversary, mathematicians at the University of Manitoba introduced the notion of "Canada perfect number". A composite integer n is called a *Canada perfect number* if the sum of the square of its digits is equal to the sum of its proper divisors > 1. In other words, n is "Canada perfect" if and only if

$$(*) \qquad \sum_{1\le i\le c(n)} \ell_i^2 = \sum_{\substack{d|n\\1<d<n}} d,$$

where $\ell_1, \ell_2, \ldots, \ell_{c(n)}$ are the digits appearing in the decimal representation of n and where $c(n)$ is the number of digits of n. One easily checks that 125 is "Canada perfect", since

$$1^2 + 2^2 + 5^2 = 30 = 5 + 25.$$

Show that the only Canada perfect numbers are 125, 581, 8549 and 16999:
(a) by using a computer to identify all Canada perfect numbers $\leq 10^6$,
(b) by proving that no Canada perfect number larger that 10^6 exists.

9. Solving Equations Involving Arithmetical Functions

(701) Using a computer,
(a) find all values of $n < 10000$ for which $4\tau(n+2) = \phi(n)$.
(b) write a program that gives the positive integers $1 \le n \le 2000$, for which $\sigma(n) = 2n - 1$.

(702) Without using a computer, find at least six solutions of $\phi(\sigma(n)) = n$.

(703) Show how one can obtain from each solution of $\phi(\sigma(n)) = n$ a corresponding solution of the equation $\sigma(\phi(n)) = n$, and then use this argument to find six solutions of $\sigma(\phi(n)) = n$ with the help of Problem 702.

(704) Show that if p and $(3^p - 1)/2$ are two prime numbers, then $n = 3^{p-1}$ is a solution of $\sigma(\phi(n)) = \phi(\sigma(n))$. Use a computer to obtain explicitly three of these solutions. Are there any other solutions besides those of this particular type?

(705) Show that the equation $\phi(\tau(n)) = \tau(\phi(n))$ has infinitely many solutions.

(706) Find all the solutions of $\tau(\gamma(n)) = \gamma(\tau(n))$, where $\gamma(n) := \prod_{p|n} p$.

(707) Consider the arithmetical function δ defined by $\delta(1) = 1$ and, for $n \ge 2$, by $\delta(n) = \prod_{p\|n} p$. In particular, if n is squarefree, we have $\delta(n) = n$. Use a computer to obtain the three smallest nonsquarefree solutions n of

$$(*) \qquad \delta(n+1) - \delta(n) = 1,$$

and then prove that the equation $(*)$ has infinitely many nonsquarefree solutions.

(708) Equation $\gamma(\sigma(n)) = n$ has only two solutions. What are they?

(709) Let k be a positive integer which is not a multiple of 8. Show that the only possible values of the smallest positive integer n which divides $\sigma_k(n)$ are 6, 10 and 34.

(710) Show that all the solutions of the equation $\dfrac{\phi(n)}{n} = \dfrac{2}{3}$ are of the form $n = 3^k$, $k = 1, 2, \ldots$.

(711) Prove that a positive integer n is a solution of the equation

$$\frac{\phi(n)}{n} = \frac{4}{7} \Longleftrightarrow n = 3^\alpha 7^\beta \text{ with } \alpha = 1, 2, \ldots, \ \beta = 1, 2, \ldots .$$

(712) Let $S(n) = \sum_{d|n} \tau(d)$. Determine all values of n such that $n = S(n)$.

(713) Find all positive integers n such that
(a) $\sigma(n) = 24$; (b) $\sigma(n) = 57$.

(714) What is the smallest positive integer n such that $\sigma(x) = n$ has exactly one solution?

(715) What is the smallest positive integer n such that $\sigma(x) = n$ has exactly two solutions?

(716) What is the smallest positive integer n such that $\sigma(x) = n$ has exactly three solutions?

(717) Let n be a fixed positive integer. Is the number of solutions of the equation $\sigma(x) = n$ finite or infinite? What about the equation $\tau(x) = n$?

(718) Is it true that n is prime if and only if $\sigma(n) = n + 1$?

(719) Let a be a rational number $\geq \frac{35}{16}$ and let n be an odd solution of the equation $\sigma(n) = an$. Show that n has at least four distinct prime factors.

(720) Let a be a rational number $\geq \frac{15}{4}$ and let n be an arbitrary solution of the equation $\sigma(n) = an$. Show that n has at least four distinct prime factors.

(721) Find two integers n for which

$$\frac{\sigma(n)}{n} = \frac{9}{4}.$$

(722) Show that there exist infinitely many positive integers m such that equation $\sigma(n) = m$ has at least three solutions.

(723) Let $\widehat{\sigma}(n)$ be the total number of subgroups of the *dihedral* group D_n of the symmetries of the regular polygon with n sides. It is possible to show that $\widehat{\sigma}(n) = \tau(n) + \sigma(n)$ (see S. Cavior [**5**]). A number n is said to be *dihedral perfect* if $\widehat{\sigma}(n) = 2n$. Characterize all such numbers which are also of the form $n = 2^k p$, where p is prime and k is a positive integer. Use a computer to find the five smallest dihedral perfect numbers of this form.

(724) Find all the solutions x of the equation $\phi(x) = 24$.

(725) Show that if $\phi(x) = 2^r N$, where $(2, N) = 1$, then x has at most r distinct odd prime factors.

(726) Find all positive integers n such that $4 \nmid \phi(n)$.

(727) Show that if $m = 2 \cdot 3^{6k+1}$ with $k \geq 1$, then

$$\phi(n) = m \iff n = 3^{6k+2} \text{ or } n = 2 \cdot 3^{6k+2}.$$

Use this to show that there exist infinitely many positive integers m such that $\#\{n : \phi(n) = m\} = 2$.

(728) Show that there does not exist any positive integer n such that $\phi(n) = 2 \cdot 7^m$, where $m \geq 1$.

(729) Let $n \geq 2$. Show that $\phi(n) = n - 1$ if and only if n is prime.

(730) Let p be a prime number such that $2p+1$ is composite. Show that $\phi(x) = 2p$ has no solutions.

(731) Show that $\phi(n) = n/2$ if and only if $n = 2^k$, for a certain integer $k \geq 1$.

(732) Show that $\phi(n) = 2n/5$ if and only if $n = 2^r 5^s$, $r, s \in \mathbb{N}$.

(733) Show that there exist infinitely many positive integers n such that $\phi(n) = n/3$.

(734) Are there any positive integers n such that $\phi(n) = n/4$?

(735) Let $n, a \in \mathbb{N}$ and let p be a prime number. Show that $\phi(p^a) = 2(6n + 1)$ if and only if $p > 6$, $p \equiv 11 \pmod{12}$ and a is even.

(736) Let $n \in \mathbb{N}$.
  (a) Show that $\frac{1}{2}\sqrt{n} \leq \phi(n) \leq n$.
  (b) Show that the equation $\phi(x) = n$ has only a finite number of integer solutions x.

(737) Find the smallest positive integer n such that $\phi(x) = n$ has no solutions. Find the smallest positive integer n such that $\phi(x) = n$ has exactly one solution, and finally find the smallest positive integer n such that $\phi(x) = n$ has exactly two solutions.

(738) Show that if a certain arithmetical function f satisfies

$$\frac{1}{\tau(n)} \sum_{d|n} f(d) = f(n)$$

for each positive integer n, then necessarily there exists a constant c such that $f(n) = c$ for each $n \in \mathbb{N}$.

(739) Show that if a certain multiplicative function f satisfies

$$\frac{1}{\tau(n)} \sum_{d|n} f(d) = f(n)$$

for each positive integer n, then necessarily $f(n) = 1$ for each $n \geq 1$.

(740) Show that the equation $(*)$ $\Omega(n)^{\Omega(n)} = n$ has infinitely many solutions.

(741) Consider the equation $(*)$ $\sum_{d|n} \gamma(d) = n$, where $\gamma(1) = 1$ and, for $n \geq 2$, $\gamma(n) = \prod_{p|n} p$. Show that the only solution $n > 1$ of $(*)$ is $n = 56$.

(742) Show that the equation $\sigma(n) - \phi(n) = (-1)^n \tau(n)$ has only one solution.

(743) Find all pairs of positive integers m and n such that

$$\phi(mn) = \phi(m) + \phi(n).$$

(744) Show that the only solutions of $\phi(n) = \gamma(n)$ are $n = 1, 4, 18$.

(745) Show that the only solutions of $\phi(n) = \gamma(n)^2$ are $n =$ 1, 8, 108, 250, 6174 and 41154.

(746) Consider the equation $(*)$ $\sigma(n) = \gamma(n)^2$. Show that each solution $n > 1$ of $(*)$ must satisfy the following properties:

(a) n is an even number.

(b) n cannot be squarefree.

Then, use a computer to find the only solution $n > 1$ of $(*)$ which is smaller than 10^8.

(747) Let k be an arbitrary positive integer. Prove that there exists infinitely many positive integers n such that $\gamma(n)^k$ divides $\sigma(n)$.

(748) Show that the equation $\phi(n) + \gamma(n) = \sigma(n)$ has only one solution.

(749) Show that $\dfrac{\phi(n) + \sigma(n)}{\gamma(n)^2}$ is an integer for infinitely many positive integers n.

(750) Find all positive integers n such that $2^{\phi(n)} \leq 2n$.

10. Special Numbers

(751) Squaring 12 gives 144. By reversing the digits of 144, we notice that 441 is also a perfect square. Using computer software, write a program to find all those integers n, $1 \leq n \leq N$, verifying this property.

(752) A positive integer which is divisible by the sum of its digits is called a *Niven number.* For example, 81 is a Niven number since it divisible by $8+1 = 9$; but 71 is not a Niven number since it is not divisible by $7+1 = 8$. Using computer software, write a program which finds all Niven numbers $n \in [12476, 12645]$.

(753) A positive integer is said to be a *palindrome* if by reversing the order of its digits, we obtain the same number, such as is the case with the number 12321. Use a computer to show that 26 is the smallest positive integer which is not a palindrome, but such that its square is a palindrome. Find other integers having this property.

(754) A positive integer N is called a *Cullen number* if it is of the form $n \cdot 2^n + 1$, $n > 1$. Find the Cullen prime numbers smaller than 1000.

(755) Write a program which allows one to find the positive integers $\leq N$ which can be written as the sum of two squares. Use this program to determine all the positive integers ≤ 300 with this property.

(756) Carmichael's conjecture states that for each positive integer n, there exists an integer $m \neq n$ such that $\phi(m) = \phi(n)$, where ϕ stands for Euler's function (see Schlafly and Wagon [**36**]). Write a program which verifies this conjecture for a given integer n.

(757) A positive integer N is called a *Silverbach number* if it can be written as the sum of two prime numbers in three different ways. Using computer software, write a program which allows one to write any integer n as the sum of two prime numbers in one way, in two distinct ways, in three distinct ways, and so on.

(758) A prime number p is called a *Wilson prime* if $(p-1)! \equiv -1 \pmod{p^2}$. Using a computer, find the three smallest Wilson primes.

(759) Let $k \geq 1$ be an integer. A positive integer n is said to be *k–hyperperfect* if

$$n = 1 + k \sum_{\substack{d|n \\ 1<d<n}} d.$$

A 1-hyperperfect number is simply a perfect number.

(a) Show that a positive integer n is k-hyperperfect if and only if $k\sigma(n) = (k+1)n + k - 1$.

(b) Show that a positive integer n is k-hyperperfect if and only if $\sigma(n) = n + 1 + \frac{n-1}{k}$.

(c) Show that if n is k-hyperperfect, then $n \equiv 1 \pmod{k}$.

(d) Show that if n is k-hyperperfect, then the smallest prime factor of n is larger than k.

(e) Prove that no prime power can be a k-hyperperfect number, for any integer $k \geq 1$.

(f) Use a computer to find all 2-hyperperfect numbers smaller than 10^6.

(g) Construct an algorithm which allows one to identify all 2-hyperperfect numbers $< 10^9$ of the form $3^\alpha \cdot p$, where α is a positive integer and where $p > 3$ is a prime number.

(760) Show that if we add the digits of an even perfect number larger than 6 and we then add the digits of the number thus obtained, and so on until we obtain a one-digit number, then this digit must be 1.

(761) Show that if $\{t_k\}$ stands for the increasing sequence of triangular numbers, then, for each positive integer n,

$$\sum_{k=1}^{n} t_k = \frac{n(n+1)(n+2)}{6}.$$

(762) Show that the Catalan number $\dfrac{(2n)!}{n!(n+1)!}$ is an integer for each integer $n \geq 0$.

(763) Show that the following method, invented by Thabit ben Korrah (826–901), an Arabic mathematician of the ninth century, for finding amicable numbers does work: if $p = 3 \cdot 2^{k-1} - 1$, $q = 3 \cdot 2^k - 1$ and $r = 9 \cdot 2^{2k-1} - 1$ are primes for a certain positive integer k, then the numbers

$$M = 2^k pq \quad \text{and} \quad N = 2^k r$$

form an amicable pair.

(764) Show that the quotient of two triangular numbers can never be 4.

(765) A positive integer n is said to be *abundant* if $\sigma(n) > 2n$. Use a computer to find the smallest odd abundant number, and then prove that there exist infinitely many abundant numbers.

(766) Let n be a positive integer. Show that $\dfrac{\sigma(n)}{n} \geq \dfrac{\sigma(d)}{d}$ for each divisor d of n. Use this result to show that a positive integer n which is a multiple of 6 is a nondeficient number, that is such that $\sigma(n) \geq 2n$.

(767) Show that there exist infinitely many positive integers n such that $n|(2^n + 1)$.

(768) Show that if n is an integer larger than 1 such that $n|(2^n + 1)$, then n is a multiple of 3.

(769) Prove that a Fermat number $F_m = 2^{2^m} + 1$ cannot be equal to p^k, where p is prime and k is an integer ≥ 2.

(770) Does there exist a prime number p which is a factor of two Mersenne numbers (that is numbers of the form $2^q - 1$, where q is a prime number)?

(771) Use a computer to find the two smallest nondeficient consecutive numbers; that is find the smallest number n such that $\sigma(n-1)/(n-1) \geq 2$ and $\sigma(n)/n \geq 2$. Proceed in the same manner to find the three smallest nondeficient consecutive numbers. Finally, show that given an arbitrary integer $k \geq 2$, there exist k nondeficient consecutive numbers.

(772) Show that there exists a positive integer n such that $\sigma(n) \geq 3n$ and $\sigma(n+1) \geq 3(n+1)$.

(773) Show that each odd tri-perfect number must have at least eight distinct prime factors.

(774) Show that if a and b are two positive integers such that $ab+1$ is a perfect square, then the set

$$A = \{a, b, a + b + 2\sqrt{ab+1}, 4(a + \sqrt{ab+1})(b + \sqrt{ab+1})\sqrt{ab+1}\}$$

is such that if $x, y \in A$, $x \neq y$, then $xy+1$ is also a perfect square. Then, find two sets A with this property.

(775) Show that for each positive integer n equal to twice a triangular number, the corresponding expression

$$\sqrt{n+\sqrt{n+\sqrt{n+\sqrt{n+\ldots}}}}$$

represents an integer.

(776) Prove Cassiny's identity

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n \qquad (n = 2, 3, \ldots),$$

where F_n stands for the n-th Fibonacci number.

(777) Show that the set $A = \{F_{2n}, F_{2n+2}, F_{2n+4}, 4F_{2n+1}F_{2n+2}F_{2n+3}\}$, where F_m stands for the m-th Fibonacci number, is such that if $x, y \in A$, $x \neq y$, then $xy+1$ is a perfect square.

(778) Show that, for each integer $n \geq 1$, the number

$$\frac{n(n+1)(n+2)(n+3)}{8}$$

is a triangular number.

(779) Show that there exist infinitely many prime numbers whose last four digits are 7777. Find five such primes.

(780) Use a computer to find the three smallest integers $n > 1$ which have the property of being divisible by the sum of the squares of their digits as well as by the product of the squares of their digits. Deduct the existence of a fourth one.

(781) We know that $\phi(p) = p - 1$ if p is prime. In 1932, Derrick Henry Lehmer (1905–1991) conjectured that there does not exist any composite number n such that $\phi(n)$ is a proper divisor of $n - 1$. Show that if such a number exists, it must be a Carmichael number.

(782) Let us write the integer $n > 9$ in the form $n = d_1 d_2 \cdots d_r$, where $d_1, d_2, \ldots$, d_r are the r digits of n. Show that there exist only a finite number of integers n such that

$$n = d_1^1 + d_2^2 + d_3^3 + \cdots + d_r^r$$

and use a computer to find the eight smallest such numbers $n > 9$.

(783) Let us write the integer $n > 9$ in the form $n = d_1 d_2 \cdots d_r$, where $d_1, d_2, \ldots$, d_r are the r digits of n. Show that there exists no number n such that

$$n = d_1^r + d_2^{r-1} + d_3^{r-2} + \cdots + d_r^1.$$

(784) Show that there exist infinitely many numbers n such that $\sigma(n) = 2n - 1$.

(785) Given a positive integer $n \equiv 2 \pmod 3$, show that each odd prime divisor of $n^2 + n + 1$ is congruent to 1 modulo 3.

11. Diophantine Equations

(786) For which positive integer(s) x is it true that

$$(*) \qquad x^3 + (x+1)^3 + (x+2)^3 = (x+3)^3 \ ?$$

(787) Show that the Diophantine equation

$$x^3 + 5 = 117y^3$$

has no solutions.

(788) One day, as the English mathematician Godfrey Harold Hardy (1877–1947) was visiting Srinivasa Ramanujan (1885–1920) at the hospital, the patient commented to his visitor that the number on the license plate of the taxi that had brought him, namely 1729, was a very special number: it is the smallest positive integer which can be written as the sum of two cubes in two different ways, namely

$$1729 = 1^3 + 12^3 = 9^3 + 10^3.$$

Using the identity

$$(*) \quad (3a^2 + 5ab - 5b^2)^3 + (4a^2 - 4ab + 6b^2)^3 = (-5a^2 + 5ab + 3b^2)^3 + (6a^2 - 4ab + 4b^2)^3$$

due to Ramanujan, show that there exist infinitely many positive integers which can be written as a sum of two cubes in two different ways. Does this identity allow one to find the "double" representation of 1729?

(789) Let $a, b, c \in \mathbb{Z}$. Show that $ax + by = b + c$ is solvable in integers x and y if and only if $ax + by = c$ is also solvable.

(790) Let $a, b, c \in \mathbb{Z}$. Show that $ax + by = c$ is solvable in integers x and y if and only if $(a, b) = (a, b, c)$.

(791) Let a and b be positive integers such that $(a, b) = 1$. Show that $ax+by = n$ has positive integer solutions if $n > ab$, while it has no positive integer solution if $n = ab$.

(792) Find the positive integer solution(s) of the system of equations

$$x + y + z = 100,$$
$$2x + 5y + \frac{z}{10} = 100.$$

(793) The triangle whose sides have lengths 5, 12 and 13 respectively has the property that its perimeter is equal to its area. There exist exactly five such triangles with integer sides. Which are they?

(794) Identify all the integer solutions, if any, to the equation

$$x^2 + 3y = 5.$$

(795) Show that if the positive integers x, y, z are the respective lengths of the sides of a rectangular triangle, then at least one of these three numbers is a multiple of 5.

(796) Let a, b and c be three real nonnegative numbers. Show that the system of equations

$$\begin{aligned} ax + by + cxy &= a + b + c, \\ by + cz + ayz &= a + b + c, \\ cz + ax + bzx &= a + b + c \end{aligned}$$

has one and only one solution in nonnegative integers x, y, z. What is this solution? Why is it so?

(797) Let a and b be positive integers such that $(a, b) = 1$. Show that $ax + by = ab - a - b$ has no solutions in integers $x \geq 0$ and $y \geq 0$.

(798) Let a and b be positive integers such that $(a, b) = 1$. Show that the number of nonnegative solutions of $ax + by = n$ is equal to

$$\left[\frac{n}{ab}\right] \quad \text{or} \quad \left[\frac{n}{ab}\right] + 1.$$

(799) At the fruit counter in a store, apples are sold 5 cents each and oranges are sold 7 cents each. Say Peter purchases four apples and twelve oranges. Peter notices that Paul also bought apples and oranges and that he pays the same total amount as you did, but with a different number of apples and oranges. Knowing that Paul purchased at least three oranges, does Peter have enough information to determine the exact number of apples and oranges purchased by Paul?

(800) Determine the set of solutions of the Diophantine equation $3x + 7y = 11$ located in the third quadrant of the cartesian plane.

(801) Determine the set of solutions of the Diophantine equation $5x + 7y = 11$ located above the line $y = x$.

(802) Assume that the set E of solutions of the Diophantine equation

$$(*) \qquad ax + by = 11$$

is given by

$$E = \{(x, y) : x = 5 - 4t \text{ and } y = 1 - 3t, \text{ where } t \in \mathbb{Z}\}.$$

Determine the values of a and b.

(803) Find the primitive solutions of $x^2 + 3y^2 = z^2$, that is those solutions x, y, z which have no common factor other than 1.

(804) Show that the only nonzero integer solutions (x, y, z) to the system of equations

$$x + y + z = x^3 + y^3 + z^3 = 3$$

are $(1, 1, 1)$, $(-5, 4, 4)$, $(4, -5, 4)$ and $(4, 4, -5)$.

(805) Show that the equation

$$x^2 = y^3 + z^5$$

has infinitely many solutions in positive integers x, y z.

(806) Find the four different ways of writing 136 as a sum of two positive integers, one of which is divisible by 5 and the other by 7.

(807) Any solution in positive integers x, y, z of $x^2 + y^2 = z^2$ is called a Pythagorean triple, since in such a case there exists a rectangular triangle whose sides have x, y, z for their respective lengths. Find all Pythagorean triples whose terms form an arithmetic progression.

(808) Find the dimensions of the Pythagorean triangle whose hypotenuse is of length 281.

(809) Show that 60 divides the product of the lengths of the sides of a Pythagorean triangle.

(810) Find every Pythagorean triangle whose area is equal to three times its perimeter.

(811) Find every Pythagorean triangle whose perimeter is equal to twice its area.

(812) Show that $\{x, y, z\} = \{3, 4, 5\}$ is the only solution of $x^2 + y^2 = z^2$ with consecutive integers x, y, z.

(813) Show that $n^2 + (n+1)^2 = 2m^2$ is impossible for $n, m \in \mathbb{N}$.

(814) Show that the equation $x^2 + y^2 = 4z + 7$ has no integer solution.

(815) Find all integer solutions of $x^2 + y^2 = z^4$ such that $(x, y, z) = 1$.

(816) Find all integer points on the line $x + y = 1$ which are located inside the circle centered at the origin and of radius 3.

(817) Find all primitive solutions of the Diophantine equation

$$x^2 + 3136 = z^2.$$

(818) Find all integer solutions of the equation

$$x^2 + y^2 = xy.$$

(819) Find the solutions of the Diophantine equation

$$(*) \qquad x^2 + 2y^2 = 4z^2.$$

(820) Find a triangle such that each of its sides is of integer length and for which an interior angle is equal to twice another interior angle.

(821) Find all positive integer solutions to the equation $x^2 + y^2 = 10$. Do the same for $x^2 + y^2 = 47$.

(822) Find all positive rational solutions of $x^2 + y^2 = 1$.

(823) Find all primitive Pythagorean triangles such that the length of one of their sides is equal to 24.

(824) Show that the radius of any circle inscribed in a Pythagorean triangle is an integer.

(825) Show that the equation $x^2 + y^2 + z^2 = 2239$ has no solutions in positive integers x, y, z.

(826) Show that

$$t^2 = x^2 + y^2 + z^2$$

has no nontrivial integer solution with t even and with x, y, z having no common factor.

(827) Find all primitive solutions of the Diophantine equation $x^2 + 2y^2 = z^2$.

(828) Find all positive integer solutions to the system of equations

$$\begin{cases} a^3 - b^3 - c^3 = 3abc, \\ a^2 = 2(b+c). \end{cases}$$

(829) Find all integer solutions of $y^2 + y = x^4 + x^3 + x^2 + x$.

(830) Find the smallest prime number which can be written in each of the following forms: $x^2 + y^2, x^2 + 2y^2, \dots, x^2 + 10y^2$.

(831) Determine the set of quadruples (x, y, x, w) verifying $x^3 + y^3 + z^3 = w^3$ and such that x, y, z and w are positive integers in arithmetical progression.

(832) Consider the sequence $8, 26, 56, 98, 152, \ldots$, that is the sequence $\{x_n\}$ defined by $x_1 = 8$ and $x_{n+1} = x_n + 6(2n+1)$, $n \geq 1$, and show that for $n > 1$, x_n cannot be the cube of an integer.

(833) Show that $x^n + 1 = y^{n+1}$ has no solutions in positive integers x, y, n $(n \geq 2)$ with $(x, n+1) = 1$.

(834) Show that neither of the equations

$$3^a + 1 = 5^b + 7^c \quad \text{and} \quad 5^a + 1 = 3^b + 7^c$$

has a solution in integers a, b, c other than $a = b = c = 0$.

(835) Find all integer triples (x, y, z) such that $4^x + 4^y + 4^z$ is a perfect square.

(836) Show that there exist solutions in positive integers a, b, c, x, y to the system of equations

$$\begin{cases} a + b + c = x + y, \\ a^3 + b^3 + c^3 = x^3 + y^3. \end{cases}$$

Show, in particular, that there exist infinitely many solutions such that a, b, c are in arithmetic progression.

(837) Solve each of the following Diophantine equations: (here m is a nonnegative integer)

$$x^m(x^2 + y) = y^{m+1},$$
$$x^m(x^2 + y^2) = y^{m+1}.$$

(838) Can the following equations be verified for an appropriate choice of integers x, a, b, c, d?

$$(x+1)^2 + a^2 = (x+2)^2 + b^2 = (x+3)^2 + c^2 = (x+4)^2 + d^2.$$

(839) Does the equation

$$x^2 + y^2 + z^2 = xyz - 1$$

have integer solutions?

(840) Find all pairs of real numbers (x, y) which satisfy the two equations:

$$(*) \qquad 2x^3 - x^2 + y^2 = 1,$$
$$(**) \qquad 2y^3 - y^2 + x^2 = 1.$$

(841) Find all positive integer solutions x, y of $x^y = y^{x-y}$.

(842) Find all integers solutions x, y, z to the system of equations

$$\begin{cases} 2x(1 + y + y^2) = 3(1 + y^4), \\ 2y(1 + z + z^2) = 3(1 + z^4), \\ 2z(1 + x + x^2) = 3(1 + x^4). \end{cases}$$

(843) Prove that there exist infinitely many integers a, b, c, d such that $a > b > c > d > 1$ and $a!\,d! = b!\,c!$.

(844) Show that the equation $x^3 + y^3 + z^3 = 4$ has no solutions in integers. What about the equation $x^3 + y^3 + z^3 = 5$?

(845) Does the Diophantine equation $x^4 = 4y^2 + 4y - 80$ have any solutions? If so, what are they? If no, explain why.

(846) Does the Diophantine equation $x^4 + y^4 + z^4 = 363932239$ have any solutions? If so, what are they? If no, explain why.

(847) Let a be an arbitrary integer. Does the Diophantine equation

$$303x + 57y = a^2 + 1$$

have any solution?

(848) Does the Diophantine equation
$$x^4 = 4y^2 + 4y - 15$$
have any solution?

(849) Do integers x, y, z exist such that
$$x^4 + (2y+1)^4 = z^2?$$

(850) Determine the set of positive solutions of the Diophantine equation
$$x^2 = y^4 + 8.$$

(851) Let p be an odd prime number. Assume that $q = p + 8$ is also a prime number. Analyze the set of solutions of the Diophantine equation
$$x^2 = y^4 + pq$$
and give one such solution explicitly.

(852) Does the Diophantine equation
$$x^2 + y^2 + 2x + 4y + 4z + 2 = 0$$
have any solution?

(853) Does the Diophantine equation
$$x^4 + y^4 + z^4 + u^4 = 3xyzu$$
have any nonzero solution?

(854) Does the Diophantine equation
$$x^3 + 2y^3 = 4z^3$$
have any nonzero solution?

(855) Find all integer solutions of $x^2 + y^2 = 8z + 7$.

(856) Show that $x^4 + y^4 = 7z^2$ has no solutions in $\mathbb{N}$. What about the equation $x^4 + y^4 = 5z^2$?

(857) Does the equation $x^4 + x^2 = y^4 + 5$ have any solution in integers x and y?

(858) Let $0 < x < y < z$ be integers such that $x^2 + y^2 = z^2$. Show that for each integer $n > 2$, $x^n + y^n = z^n$ is impossible.

(859) Prove that the equation
$$x^3 + 3y^3 = 9z^3$$
has no nontrivial integer solution.

(860) Let p be a prime number. Does the Diophantine equation
$$x^4 + py^4 + p^2z^4 = p^3w^4$$
have any trivial solution?

(861) Show that
$$x^2 + y^2 + z^2 = 2xyz$$
has no nontrivial integer solution.

(862) Determine all rational solutions of the equation
$$x^3 + y^3 = x^2 + y^2.$$

(863) Show that the Diophantine equation
$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} + \frac{1}{x_1 x_2 \cdots x_n} = 1$$
has at least one solution for each integer $n \geq 1$.

(864) Show that the Diophantine equation

$$x^2 + y^2 + z^2 = x^2y^2$$

has no nontrivial solution.

(865) Show that the equation

$$x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 1\,234\,567\,891\,314$$

has no integer solution.

(866) Prove that the Diophantine equation

$$(x + y)^2 + (x + z)^2 = (y + z)^2$$

has no solutions in odd integers x, y, z.

(867) Let p be a fixed prime number. Find all positive integer solutions of $x^2 + py^2 = z^2$.

(868) Show that there exist infinitely many solutions to the Diophantine equation $x^2 + 4y^2 = z^3$.

(869) Find all solutions, for x, y integers and n positive integers, to the Diophantine equation $x^n + y^n = xy$.

(870) Show that the equation $n^x + n^y = n^z$ has positive integer solutions only if $n = 2$.

(871) Show that the equation $n^x + n^y + n^w = n^z$ has positive integer solutions only if $n = 2$ or 3.

(872) Show that the *abc* conjecture implies the following result: The equation $x^p + y^q = z^r$ has no solutions in positive integers p, q, r, x, y, z with $z \geq z_0$ and

$$(*) \qquad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1,$$

so that in particular the Fermat equation $x^n + y^n = z^n$ has no nontrivial solution for $n \geq 4$ and z sufficiently large.

(873) Show that if the *abc* conjecture is true, then there can exist only a finite number of triples of consecutive powerful numbers.

(874) Show that if the *abc* conjecture is true, then there exist only a finite number of positive integers n such that $n^3 + 1$ is a powerful number. Moreover, find two numbers n with this property.

(875) Erdős conjectured that the equation $x + y = z$ has only a finite number of solutions in 4-powerful integers x, y, z pairwise coprime. Show that the *abc* conjecture implies this conjecture.

(876) Show that if the *abc* conjecture is true, then there exist only a finite number of 4-powerful numbers which can be written as the sum of two 3-powerful numbers pairwise coprime.

(877) Given an integer $n \geq 2$, let $P(n)$ stand for the largest prime factor of n. Prove that it follows from the *abc* conjecture that, for each real number $y > 0$, the set $A_y := \{p \text{ prime} : P(p^2 - 1) \leq y\}$ is a finite set and therefore has a largest element $p = p(y)$.

(878) In 1877, Edouard Lucas (1842–1891) observed that although 2701 is a composite number, we have that $2^{2700} \equiv 1 \pmod{2701}$, thus providing a counter-example to the reverse of Fermat's Little Theorem. More generally, show that one can construct a large family of such counter-examples

by considering the numbers $n = pq$, where p and q are prime numbers such that $p \equiv 1 \pmod 4$ and $q = 2p - 1$.

(879) Show that if the *abc* conjecture is true, then for any $\varepsilon > 0$, there exists a positive constant $M = M(\varepsilon)$ such that for all triples (x_1, x_2, x_3) of positive integers, pairwise coprime and verifying $x_1 + x_2 = x_3$, we have that

$$(*) \qquad \min(x_1, x_2, x_3) \le M\,(\gamma(x_i))^{3+\varepsilon} \qquad (i = 1, 2, 3).$$

(880) In 1979, Enrico Bombieri naively claimed that: "the equation

$$\binom{x}{n} + \binom{y}{n} = \binom{z}{n} \qquad (n \ge 3)$$

had no solutions in positive integers x, y, z." Was Bombieri right? If so, prove it; if no, provide a counter-example.

(881) Let p be an odd prime number and let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be positive integers not exceeding $p - 1$. Show that the Diophantine equation

$$n^p = x_1^{\alpha_1} + x_2^{\alpha_2} + \cdots + x_r^{\alpha_r}$$

has solutions in positive integers $n, x_1, x_2, \ldots, x_r$.

(882) Even though, according to Fermat's Last Theorem, for each prime number $p \ge 3$, the equation $x^p + y^p = z^p$ has no solutions in positive integers x, y, z, show that the equation $x^{p-1} + y^{p-1} = z^p$ always has solutions (besides the trivial one $x = y = z = 2$).

12. Quadratic Reciprocity

(883) Characterize all prime numbers $p > 11$ for which

$$x^2 \equiv 11 \pmod{p}$$

has a solution.

(884) Which of the following congruences have solutions?
  (a) $x^2 \equiv 1 \pmod{3}$;
  (b) $x^2 \equiv -1 \pmod{3}$;
  (c) $x^2 + 4x + 8 \equiv 0 \pmod{3}$;
  (d) $x^2 + 8x + 16 \equiv -1 \pmod{17}$.

(885) Find the solutions of the congruence $2x^2 + 3x + 1 \equiv 0 \pmod{7}$.

(886) Show that $(1!)^2 + (2!)^2 + \cdots + (n!)^2$ is never a perfect square, whatever the integer $n > 1$.

(887) Let $n \in \mathbb{N}$. Show that the odd prime divisors of $n^2 + 1$ are of the form $12k + 1$ or of the form $12k + 5$.

(888) Let $p > 3$ be a prime number. Show that p divides the sum

$$\sum_{\substack{j=1 \\ \left(\frac{j}{p}\right)=1}}^{p-1} j.$$

(889) Assuming that m is a positive integer such that $p = 4m+3$ and $q = 2p+1$ are two prime numbers, show that $q | M_p = 2^p - 1$. Use this result to show that the Mersenne number $M_{1\,122\,659}$ is composite.

(890) Show that 9239 divides $2^{4619} - 1$.

(891) Show that 5 is a nonquadratic residue of all the prime numbers of the form $6^n + 1$.

(892) Does there exist a perfect square of the form $1997k - 1$?

(893) Show that there exist infinitely many prime numbers of the form $3k + 1$.

(894) Does there exist a perfect square of the form $1! + 2! + \cdots + k!$ with $k > 3$?

(895) Show that for each integer $n > 1$, $(2^n - 1) \nmid (3^n - 1)$.

(896) Let p and q be two odd prime numbers, and a an integer. If $p = q + 4a$, is it true that $\left(\frac{p}{q}\right) = \left(\frac{a}{q}\right)$?

(897) If p is a prime number of the form $24k + 1$, is it true that $\left(\frac{3}{p}\right) = 1$?

(898) Does the congruence $x^2 \equiv 52 \pmod{159}$ have any solutions?

(899) If p is a prime number of the form $8k+3$ and if $q = \frac{p-1}{2}$ is a prime number, can one conclude that q is a quadratic residue modulo p?

(900) Show that 3 is a nonquadratic residue of all Mersenne primes larger than 3.

(901) If p is a prime number of the form $p = 8k + 7$, show that

$$p | 2^{\frac{p-1}{2}} - 1.$$

(902) Does the congruence $x^2 \equiv 2 \pmod{231}$ have any solution? If so, what are they? If not, explain why.

(903) Does there exist a positive integer n and a prime number p of the form $p = 100k + 3$ such that $p | n^2 + 1$? Explain.

(904) Is it true that there exist infinitely many positive integers n such that $23|n^2+14n+47$? Explain.

(905) Does there exist an integer x such that the prime number 541 divides x^2-3x-1? Explain.

(906) If p is a prime number, $p\equiv 1 \pmod{24}$, does the congruence $x^2\equiv 6 \pmod p$ have any solution? Explain.

(907) Let n be a positive integer such that $p=4^n+1$ is a prime number. Does the congruence $x^2\equiv 3 \pmod p$ have any solution? Explain.

(908) Let A be the set of integers a, $1\le a\le 43$, for which there exists a prime number $p\equiv a \pmod{44}$ such that the corresponding congruence

$$x^2\equiv 11 \pmod p$$

has solutions. Determine A.

(909) Find all prime numbers p for which $\left(\frac{5}{p}\right)=-1$.

(910) Let p and q be odd prime numbers such that $p=q+4a$, $a\in\mathbb{N}$. Show that

$$\left(\frac{a}{p}\right)=\left(\frac{a}{q}\right).$$

(911) Of which prime numbers is the number -2 a quadratic residue?

(912) Let p be an odd prime number. Show that $x^2\equiv 2 \pmod p$ has solutions if and only if $p\equiv 1$ or $7 \pmod 8$. Using this result, prove that $2^{4n+3}\equiv 1 \pmod{8n+7}$ for each integer $n\ge 0$. In particular, find a proper divisor of the Mersenne number $2^{131}-1$.

(913) Observing that $2717=11\cdot 13\cdot 19$, determine if the quadratic congruence $x^2\equiv 1237 \pmod{2717}$ has solutions.

(914) Let a be an integer such that $(a,p)=1$. Determine all prime numbers p such that $\left(\frac{a}{p}\right)=\left(\frac{p-a}{p}\right)$.

(915) Does the congruence $x^2\equiv 131313 \pmod{1987}$ have any solutions?

(916) Show that the equation $x^2-y^3=7$ has no integer solution.

(917) Determine all prime numbers p for which 15 is a quadratic residue modulo p.

(918) Show that the statement of the law of quadratic reciprocity can be written (as Gauss did) as

$$\left(\frac{p}{q}\right)=\left(\frac{(-1)^{(q-1)/2}q}{p}\right).$$

(919) Does the congruence $x^2\equiv 34561 \pmod{1234577}$ have any solution?

(920) Show that if r is a quadratic residue modulo $m>2$, then

$$r^{\phi(m)/2}\equiv 1 \pmod m.$$

(921) Let a be an integer >1 and let n be a positive integer. Show that $n|\phi(a^n-1)$.

(922) Show that if p is an odd prime number, then

$$\sum_{j=1}^{p-1}\left(\frac{j}{p}\right)=0.$$

(923) Let p be an odd prime number. Show that

$$\sum_{k=1}^{p-2}\left(\frac{k(k+1)}{p}\right)=-1.$$

(924) Let $p>5$ be a prime number. Using the Problem 923, show that one can always find two consecutive integers which are quadratic residues modulo p, as well as two consecutive integers which are quadratic nonresidues modulo p.

(925) Find all prime numbers p such that the corresponding numbers $5p+1$ are perfect squares. Is it possible to find prime numbers p for which $5p+2$ are perfect squares?

(926) Let $f\colon \mathbb{Z}\to\mathbb{Z}$ be a polynomial function and let a, b be integers. Set $\left(\frac{m}{p}\right)=0$ if $p|m$. If $(a,p)=1$, show that

$$\sum_{k=0}^{p-1}\left(\frac{f(ak+b)}{p}\right)=\sum_{k=0}^{p-1}\left(\frac{f(k)}{p}\right).$$

Use this to prove that if $(a,p)=1$, then

$$\sum_{k=0}^{p-1}\left(\frac{ak+b}{p}\right)=0.$$

(927) Let $a,b\in\{-1,1\}$, p be an odd prime number and

$$N(a,b)=\#\left\{k \mid 1\le k\le p-2,\quad \left(\frac{k}{p}\right)=a,\quad \left(\frac{k+1}{p}\right)=b\right\}.$$

Show that

$$N(a,b)=\frac{1}{4}\left(p-2-b-ab-a(-1)^{(p-1)/2}\right).$$

Use this to prove that the number of pairs of consecutive quadratic residues modulo p is given by

$$N(1,1)=\frac{p-4-(-1)^{(p-1)/2}}{4}.$$

(928) Let p be a prime number satisfying $p\equiv 1 \pmod 4$. Show that

$$\sum_{j=1}^{(p-1)/2}\left(\frac{j}{p}\right)=0.$$

(929) Let p be a prime number such that $p\equiv 1 \pmod 4$. Show that

$$\sum_{k=1}^{p-1}k\left(\frac{k}{p}\right)=0.$$

(930) Let p be a prime number such that $p\equiv 1 \pmod 4$. Show that

$$\sum_{\substack{k=1\\ \left(\frac{k}{p}\right)=1}}^{p-1} k=\frac{p(p-1)}{4}.$$

(931) Let p be a prime number such that $p \equiv 3 \pmod 4$. Show that

$$\sum_{k=1}^{p-1} k^2 \left(\frac{k}{p}\right) = p \sum_{k=1}^{p-1} k \left(\frac{k}{p}\right).$$

(932) Show that the equation $x^2 - 33y^2 = 5$ has no integer solutions.

13. Continued Fractions

(933) Express each of the numbers $\sqrt{2}$ and $\sqrt{2}/2$ as a simple infinite continued fraction.

(934) Using continued fractions, find a solution of the equation $12x + 5y = 13$; do the same for $13x - 19y = 1$.

(935) Find the irrational number represented by the infinite continued fraction $[3, \overline{1, 4}]$.

(936) Let α be an irrational number > 1 whose representation as a simple infinite continued fraction is $[a_1, a_2, \ldots]$. Express $1/\alpha$ as a simple infinite continued fraction.

(937) Find a rational number which provides a good approximation of $\sqrt{5}$; that is, find a rational number a/b such that

$$|\sqrt{5} - a/b| < 10^{-4}.$$

(938) Let $\{p_n\}$ and $\{q_n\}$ be the sequences defined in Definition 24. Show that

$$\frac{p_n}{p_{n-1}} = [a_n, a_{n-1}, \ldots, a_1],\ n \geq 1, \qquad \frac{q_n}{q_{n-1}} = [a_n, a_{n-1}, \ldots, a_2],\ n \geq 2.$$

(939) Show that if $ax^2 - bx - c = 0$ where $abc \neq 0$ and $b^2 + 4ac$ is not a perfect square, then the continued fraction $\left[\overline{\frac{b}{a}, \frac{b}{c}}\right]$ is a real root of the quadratic equation.

(940) Find an approximation for the real roots of $2x^2 - 5x - 4 = 0$ which is accurate up to the first decimal.

(941) For each $n \in \mathbb{N}$, show that $\sqrt{n^2+1} = [n, \overline{2n}]$.

(942) Given an integer $n > 1$, show that the continued fraction which represents $\sqrt{n^2-1}$ is $[n-1, \overline{1, 2n-2}]$.

(943) For each $n \in \mathbb{N}$, show that $\sqrt{n^2+2} = [n, \overline{n, 2n}]$.

(944) Given an integer $n > 1$, show that the continued fraction that represents $\sqrt{n^2-2}$ is $[n-1, \overline{1, n-2, 1, 2n-2}]$.

(945) Find the continued fraction of $\sqrt{38}$, that of $\sqrt{47}$ and that of $\sqrt{120}$.

(946) If n is a positive integer, show that the continued fraction which represents $\sqrt{n^2+n}$ is $[n, \overline{2, 2n}]$.

(947) Given an integer $n > 1$, show that the continued fraction which represents $\sqrt{n^2-n}$ is $[n-1, \overline{2, 2n-2}]$.

(948) Given an integer $n > 1$, show that the continued fraction which represents $\sqrt{9n^2+3}$ is $[3n, \overline{2n, 6n}]$.

(949) Find the real number r whose expansion in a continued fraction is $q = [1, \overline{1, 2}]$ by multiplying the quantities $q+1$ and $q-1$.

(950) Find the best rational approximation a/b of π when $b < 1000$. Do the same for e and then for $\sqrt{5}$.

(951) Find an approximation of the irrational number $[1, 2, 3, 4, 5, 6, 7, \ldots]$ correct up to the sixth decimal.

(952) Knowing that

$$e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, \ldots],$$

find a rational number which is a correct approximation of the number e up to the fourth decimal.

(953) Knowing that

$$\pi = [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, \ldots],$$

find a rational number which is a correct approximation of the number π up to the sixth decimal.

(954) Find an approximation correct up to 10^{-4} of the number

$$[4, \overline{2, 1, 3, 1, 2, 8}].$$

(955) Let $k \geq 1$ and let $C_k = p_k/q_k$ be the k–th convergent of the irrational number α. Assume that a and b are integers, with b positive. Show that if $|\alpha - a/b| < |\alpha - p_k/q_k|$, then $b > q_{k+1}/2$.

(956) Assume that $|\sqrt{3} - a/b| < |\sqrt{3} - 26/15|$, where $b > 0$. Show that $b \geq 21$.

(957) Let $k \geq 1$ and let $C_k = p_k/q_k$ be the k–th convergent of the irrational number α. If a and b are integers with $a \geq 1$, $b \geq 1$ and

$$\alpha < \frac{a}{b} < \frac{p_k}{q_k},$$

show that $b > q_{k+1}$.

(958) Let a and b be positive integers such that $\sqrt{3} < a/b < 26/15$. Show that $b > 41$.

(959) Let a/b be a rational number such that $a/b \neq 333/106$. If $0 \leq b \leq 56$, show that $|\pi - 333/106| < |\pi - a/b|$, and thus that $333/106$ is a better approximation of π then any other rational number whose denominator is smaller or equal to 56.

(960) Let $[a_1, a_2, \ldots]$ be a simple continued fraction. Show that

$$q_n \geq 2^{(n-1)/2}, \qquad \text{for } n \geq 3.$$

(961) Show that for $n \geq 4$,

$$\begin{aligned} p_n q_{n-3} - q_n p_{n-3} &= (a_n a_{n-1} + 1)(p_{n-4} q_{n-3} - q_{n-4} p_{n-3}) \\ &= (-1)^n (a_n a_{n-1} + 1). \end{aligned}$$

(962) Given an integer $n \geq 2$, let $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ be integers such that the a_j's and b_j's are positive for each $2 \leq j \leq n$. If $a_i = b_i$ for $1 \leq i < n$ and $a_n < b_n$, show that

$$\begin{aligned} [a_1, a_2, \ldots] &< [b_1, b_2, \ldots] \qquad \text{if } n \text{ is odd}, \\ [a_1, a_2, \ldots] &> [b_1, b_2, \ldots] \qquad \text{if } n \text{ is even}. \end{aligned}$$

(963) Let $a_1, a_2, \ldots, b_1, b_2, \ldots$ and $c_1, c_2, \ldots$ be integers such that a_j, b_j and c_j are positive for each $j \geq 2$. If $c_i \leq a_i \leq b_i$ for $i \geq 1$, then show that

$$[b_1, c_2, b_3, c_4, b_5, \ldots] \leq [a_1, a_2, a_3, \ldots] \leq [c_1, b_2, c_3, b_4, c_5, \ldots].$$

(964) Let a_i be integers taking the value 1 or 2. Show that if $\alpha = [a_1, a_2, \ldots]$, then

$$\frac{1 + \sqrt{3}}{2} \leq \alpha \leq 1 + \sqrt{3}.$$

(965) A complete orbit of the Earth around the Sun takes approximately 365 days, 5 hours, 48 minutes and 46 seconds. Thus, the actual length of a year exceeds 365 days by $\frac{20926}{86400}$ of a day. In 45 B.C., Julius Ceasar used the correction $1/4$, namely by adding the 29-th of February every four

years. This approximation created an error of 10 days every 1500 years. This necessitated a modification which was done by Pope Gregory XIII in 1582. Our present calendar, known as the *Gregorian calendar*, gives an additional day on each year divisible by 4, except on the years divisible by 100 but not by 400. This correction corresponds to adding 97 days (one day per bissextile year) for a period of 400 years, which is a fairly good approximation of 20926/86400. Find an even better approximation.

(966) Let $k \geq 1$ and $C_k = p_k/q_k$ be the k–th convergent of the irrational number $\alpha = [a_1, a_2, a_3, \ldots]$. Show that

$$p_k = \det \begin{pmatrix} a_1 & -1 & 0 & \ldots & 0 & 0 \\ 1 & a_2 & -1 & \ldots & 0 & 0 \\ 0 & 1 & a_3 & \ldots & 0 & 0 \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & -1 & . \\ . & . & . & \ldots & a_{k-1} & -1 \\ 0 & 0 & 0 & \ldots & 1 & a_k \end{pmatrix}.$$

Find a similar expression for q_k.

(967) A simple infinite continued fraction is said to be *periodic* if it is of the form $[a_1, a_2, \ldots, a_n, \overline{b_1, b_2, \ldots, b_m}]$. If it is of the form $[\overline{b_1, b_2, \ldots, b_m}]$, we say that it is *purely periodic*. The smallest positive integer m satisfying the above relation is called the *period* of the simple infinite continued fraction. Show that any simple continued fraction which is purely periodic must be a *quadratic irrational number* (that is an irrational number which is a root of a quadratic equation whose coefficients are integers).

(968) Show that every periodic simple continued fraction is a quadratic irrational number.

(969) Let α be an irrational root of $f(x) := ax^2 + bx + c = 0$, where a, b, c are integers. If

$$\alpha = [a_1, a_2, \ldots], \qquad \alpha_n = [a_{n+1}, a_{n+2}, \ldots] \text{ for each } n \geq 1,$$

show that α_n $(n \geq 1)$ is a root of the polynomial $A_n x^2 + B_n x + C_n = 0$, where

$$\begin{aligned} A_n &= q_n^2 f\left(\frac{p_n}{q_n}\right) = ap_n^2 + bp_n q_n + cq_n^2, \\ B_n &= 2ap_n p_{n-1} + bp_n q_{n-1} + bp_{n-1} q_n + 2cq_n q_{n-1}, \\ C_n &= q_{n-1}^2 f\left(\frac{p_{n-1}}{q_{n-1}}\right) = ap_{n-1}^2 + bp_{n-1} q_{n-1} + cq_{n-1}^2, \end{aligned}$$

and where $B_n - 4A_nC_n = b^2 - 4ac$. Use this to prove that $A_nC_n < 0$.

(970) Let α be a quadratic irrational number and write

$$\alpha = [a_1, a_2, \ldots, a_n, \alpha_n] \qquad (n = 1, 2, \ldots),$$

where

$$\alpha_n = [a_{n+1}, a_{n+2}, \ldots] \text{ for each } n \geq 1.$$

Show that there exists a finite number of quadratic polynomials with integer coefficients, say

$$\begin{gathered}A_1x^2 + B_1x + C_1,\\ A_2x^2 + B_2x + C_2,\\ \dots\\ A_Nx^2 + B_Nx + C_N,\end{gathered}$$

of which α_n is a root.

(971) Show that each quadratic irrational number has a periodic expansion as a simple continued fraction.

(972) Given any integer $D > 1$ which is not a perfect square, the following result is known: *Let $\alpha = a + b\sqrt{D}$ and $\overline{\alpha} = a - b\sqrt{D}$, where α is a quadratic irrational number. If $\alpha > 1$ and $-1 < \overline{\alpha} < 0$, then the continued fraction which represents α is a simple continued fraction which is purely periodic.* Show that this is the case for the quadratic irrational numbers $(3+\sqrt{23})/7$, $2 + \sqrt{7}$ and $(5 + \sqrt{37})/3$.

(973) If D is a positive integer which is not a perfect square, show that the continued fraction which represents $\sqrt{D}$ is a periodic continued fraction whose period begins after the first term. In particular, show that

$$\sqrt{D} = [a_1, \overline{a_2, a_3, \dots, a_n, 2a_1}].$$

14. Classification of Real Numbers

(974) Show that the sequence $[\sqrt{2}],\ [2\sqrt{2}],\ [3\sqrt{2}],\ [4\sqrt{2}], \ldots$ contains infinitely many powers of 2.

(975) Assume that $0 < r \in \mathbb{Q}$ is given as an approximation of $\sqrt{2}$. Show that the number $\frac{r+2}{r+1}$ represents an even better approximation.

(976) Consider each of the following numbers and indicate if it is rational or irrational:

$$\text{(a)}\quad \sqrt{676}; \qquad \text{(b)}\quad \sqrt{75}+\sqrt{2}.$$

(977) Let $a = 12$, $b = 245$, $c = 363$, $d = 375$. Consider each of the following numbers and indicate if it is rational or irrational:

$$\text{(a) } \sqrt{ab}; \qquad \text{(b) } \sqrt{ac}; \qquad \text{(c) } (6ad)^{1/3}; \qquad \text{(d) } \log a.$$

(978) Let $f : \mathbb{R} \to \{0,1\}$ be the *Dirichlet function* defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Show that this function has the representation

$$f(x) = \lim_{m,n\to\infty} (\cos(m!\pi x))^n.$$

(979) Show that the positive root of the equation $x^5 + x = 10$ is irrational.

(980) Let r and s be two positive integers. If the equation $x^2 + rx + s = 0$ has a root $x_0 \in \mathbb{Q}$, show that $x_0 \in \mathbb{Z}$.

(981) Let p and q be two prime numbers. For which integers m and n is the number $m\sqrt{p} + n\sqrt{q}$ an integer?

(982) Show that the number $\alpha = 0.0110101000101\ldots$, where the j-th decimal after the dot is 1 if j is prime and 0 otherwise, is an irrational number.

(983) Let p and q be two prime numbers. Show that $\sqrt{p} + \sqrt{q}$ is necessarily an irrational number.

(984) Consider the three numbers

$$\alpha = 1 + \cfrac{1}{1 + \cfrac{1}{1+\cfrac{1}{1+\ldots}}}, \qquad \beta = \sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}, \qquad \delta = \frac{1-\sqrt{5}}{2}.$$

Find the only number t satisfying the equation

$$\alpha + \beta + t\delta = 2.$$

(985) Is the number $2^{1/3} + 3^{1/3}$ an irrational number?

(986) Is the number $\log_{10} 2$ irrational?

(987) Let $0 < \xi < 1$ be a rational number. Prove that there exists only a finite number of solutions p/q to the inequality

$$\left| \frac{p}{q} - \xi \right| < \frac{1}{q^2}.$$

(988) In 1934, Gelfond and Schneider established that if α and β are algebraic, $\alpha \neq 0, 1$, and β is irrational, then α^β is transcendental. Use this result to prove that $\dfrac{\log 3}{\log 2}$ is transcendental.

(989) Let $m \in \mathbb{Q}$, $m > 0$. Prove that $m + \dfrac{1}{m}$ is an integer if and only if $m = 1$.

(990) Find the polynomial of minimal degree of which the real number $\sqrt{2}+\sqrt{7}$ is a root.

(991) Determine the roots of the polynomial $p(x) = x^3 + 2x^2 - 1$ and indicate those which are rational numbers as well as those which are irrational numbers.

(992) Let e stand for the Euler number. Is it possible to find integers a and b such that

$$e = \frac{4}{\sqrt{ae+b}}?$$

If so, find them. If not, explain why.

(993) Is it true that the interval $[\frac{7}{2}, \frac{9}{2}]$ contains at least one transcendental number? Explain.

(994) Using the fact that

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = 2,$$

show that there exist irrational numbers α and β such that α^β is rational.

(995) Show that

(a) if y is a real nonnegative number such that e^y is rational, then y is irrational;

(b) π is an irrational number.

(996) Show that $\log 2$ (the neperian logarithm of 2) is an irrational number.

(997) Show that $\dfrac{1+7^{1/3}}{2}$ is an algebraic number of degree three by finding its minimal polynomial.

(998) Show that $1 + \sqrt{2} + \sqrt{3}$ is an algebraic number of degree four by finding its minimal polynomial.

(999) Is the number $2^{1/2} + 3^{1/3}$ irrational, algebraic, transcendental? Explain.

(1000) Without using a computer, find all rational roots of the polynomial $x^5 + 39x^4 + 83x^3 + 325x^2 - 348x - 1924$.

(1001) Does there exist a rational number x such that

$$\pi^5 x + 2\pi^4 x^2 + 3\pi^3 x^3 + 4\pi^2 x^4 + 5\pi x^5 + 6 = 0\ ?$$

Explain.